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Immersed spheres of finite total curvature into manifolds

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Abstract. We prove that a sequence of possibly branched, weak immersions of the 2-sphere $\mathbb{S}^2$ into an arbitrary compact Riemannian manifold (M^m, h) with uniformly bounded area and uniformly bounded L^2-norm of the second fundamental form either collapse to a point or weakly converges as current, modulo extraction of a subsequence, to a Lipschitz mapping of $\mathbb{S}^2$ and whose image is made of a connected union of finitely many, possibly branched, weak immersions of $\mathbb{S}^2$ with finite total curvature. We prove moreover that if the sequence belongs to a class γ of $\pi_2(M^m)$, the limiting Lipschitz mapping of $\mathbb{S}^2$ realizes this class as well.

Keywords. Conformal differential geometry, weak immersions, minimal surfaces, optimal surfaces, geometric PDE, elliptic regularity theory.

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1 Introduction

Throughout the paper (M^m, h) denotes a connected Riemannian manifold and for any $x_0 \in M^m$ we denote respectively by $\pi_2(M^m, x_0)$ the *homotopy groups of based maps* form $\mathbb{S}^2$ into M^m sending the south pole to the point x_0 and by $\pi_0(C^0(\mathbb{S}^2, M^m))$ the free homotopy classes. It is well known that the group $\pi_2(M^m, x_0)$ for different points x_0 are isomorphic to each other and $\pi_2(M)$ denotes any of the $\pi_2(M^m, x_0)$ modulo isomorphisms.

Following the classical approach of Douglas and Rado for the Plateau problem, Sacks and Uhlenbeck proceeded to the minimization of the Dirichlet energy among mappings $\vec{\Phi}$ of the two sphere $\mathbb{S}^2$ into M^m

$$E(\vec{\Phi}) = \frac{1}{2} \int_{\mathbb{S}^2} |d\vec{\Phi}|_h^2 \, d\mathrm{vol}_{\mathbb{S}^2}$$

within a fixed based homotopy class in $\pi_2(M^m, x_0)$ in order to generate area minimizing, possibly branched, immersed spheres realizing this homotopy class.

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Though the paper had a major impact in mathematics (analysis, geometry, differential topology etc.) as being the funding work for the concentration compactness theory, the program of Sacks and Uhlenbeck was only partially successful. Indeed the possible loss of compactness arising in the minimization process can generate a union of immersed spheres realizing the corresponding *free homotopy class* but for which the underlying component in the *based homotopy group* $\pi_2(M^m, x_0)$ may have been forgotten.

For instance, assuming $\pi_1(M^m) = 0$ in such a way that the free homotopy classes $\pi_0(C^0(\mathbb{S}^2, M^m))$ identify[1] to the homotopy classes of $\pi_2(M^m)$, one could consider the situation where two given distinct components γ_1, γ_2 of $C^0(\mathbb{S}^2, M^m)$ possess absolute minimizer of the area among conformal immersion of $\mathbb{S}^2$ which are "far apart" in the sense that the union of the images of two such minimizers respectively in γ_1 and in γ_2 is never connected. Considering now the component $\gamma_1 + \gamma_2$ in $\pi_2(M^m)$, a minimizing sequence for the energy in this component is likely to converge in the class of currents to the union of two disconnected conformal immersions of $\mathbb{S}^2$ each realizing respectively γ_1 and γ_2. Such a disconnected limit is achieved though the minimizing sequence was made of connected objects. It is expected that, from the mapping point of view, the limit is given by the two spheres connected by a minimizing geodesic which has been lost in the current convergence process. It is very hard in the Sacks Uhlenbeck's approach to decide for which class this phenomenon indeed occurs and to identify the classes γ_1, γ_2 which are realized by minimal conformal immersions and to dissociate them from the somehow *not satisfying classes* $\gamma_1 + \gamma_2$. At least Sacks and Uhlenbeck could prove that the set of *satisfying classes* is generating the free homotopy group $\pi_2(M^m)$.

Theorem 1.1 ([23]). *There exists a set of free homotopy classes*

$$\Lambda_i \in \pi_0(C^0(\mathbb{S}^2, M^m))$$

such that the elements $\{\lambda \in \Lambda_i\}$ generate $\pi_2(M^m)$ acted on by $\pi_1(M^m)$ and each Λ_i contains a conformal branched immersion of a sphere having least area among maps of $\mathbb{S}^2$ into M^m which lie in Λ_i.

In order to remedy to the difficulty to identify which class in $\pi_2(M^m)$ is realized by branched minimal immersions, in the present work we generate an alternative representative when the given class is not achieved. To that end we will consider the minimization of the following energy:

$$L(\vec{\Phi}) := \int_{\mathbb{S}^2} \left[1 + |\vec{H}_{\vec{\Phi}}|_h^2\right] d\,\mathrm{vol}_g,$$

[1] We recall that the free homotopy classes is in one-to-one correspondence with the set of orbits of the action of $\pi_1(M^m)$ on $\pi_2(M^m, x_0)$.

among possibly branched immersions $\vec{\Phi}$, where $d\,\mathrm{vol}_g$ denotes the volume form associated to the induced metric $g := \vec{\Phi}^* h$ by $\vec{\Phi}$ on $\mathbb{S}^2$ and $\vec{H}_{\vec{\Phi}}$ is the mean curvature vector associated to the immersion.

Observe that this energy we propose to minimize is the sum of the area and the so-called *Willmore energy* of the immersion

$$L(\vec{\Phi}) := A(\vec{\Phi}) + W(\vec{\Phi}).$$

Minimizing this energy is a natural generalization of Sacks Uhlenbeck's procedure in the sense that, if a class γ in $\pi_2(M^m)$ possesses an area minimizing immersion $\vec{\Phi}$, then $\vec{H}_{\vec{\Phi}} \equiv 0$ and thus $\vec{\Phi}$ has also to minimize L in its homotopy class. Moreover, as we will see below in Theorem 1.5 and in the subsequent work [16], the minimization procedure of L has the advantage to always provide a smooth representative of any chosen class of $\pi_2(M^m)$.

Minimizing L among smooth immersions is of course a-priori an ill-posed variational problem exactly like minimizing the Dirichlet energy E right away among C^1 maps has little chance of success. In [22] (see also [20]), the second author introduced a suitable setting for dealing with minimization problems whose highest order term is given by the Willmore energy. We now recall the notion of *weak branched immersions with finite total curvature*.

By virtue of Nash's theorem we can always assume that M^m is isometrically embedded in some Euclidean space $\mathbb{R}^n$. We first define the Sobolev spaces from $\mathbb{S}^2$ into M^m as follows: for any $k \in \mathbb{N}$ and $1 \le p \le \infty$

$$W^{k,p}(\mathbb{S}^2, M^m) := \left\{ u \in W^{k,p}(\mathbb{S}^2, \mathbb{R}^n) : u(x) \in M^m \text{ for a.e. } x \in \mathbb{S}^2 \right\}.$$

We introduce the space of *possibly branched Lipschitz immersions* as follows: A map $\vec{\Phi} \in W^{1,\infty}(\mathbb{S}^2, M^m)$ is a *possibly branched Lipschitz immersion* if

(i) there exists a $C > 1$ such that, for a.e. $x \in \mathbb{S}^2$,

$$C^{-1} |d\vec{\Phi}|^2(x) \le |d\vec{\Phi} \wedge d\vec{\Phi}|(x) \le |d\vec{\Phi}|^2(x), \tag{1.1}$$

where the norms of the different tensors have been taken with respect to the standard metric on $\mathbb{S}^2$ and with respect to the metric h on M^m and where $d\vec{\Phi} \wedge d\vec{\Phi}$ is the tensor given in local coordinates on $\mathbb{S}^2$ by

$$d\vec{\Phi} \wedge d\vec{\Phi} := 2\, \partial_{x_1} \vec{\Phi} \wedge \partial_{x_2} \vec{\Phi}\, dx_1 \wedge dx_2 \in \bigwedge^2 T^*\mathbb{S}^2 \otimes \bigwedge^2 T_{\vec{\Phi}(x)} M^m,$$

(ii) there exists at most finitely many points $\{a_1, \ldots, a_N\}$ such that for any compact $K \subset \mathbb{S}^2 \setminus \{a_1, \ldots, a_N\}$

$$\operatorname*{ess\,inf}_{x \in K} |d\vec{\Phi}|(x) > 0. \tag{1.2}$$

For any *possibly branched Lipschitz immersion* we can define almost everywhere the *Gauss map*

$$\vec{n}_{\vec{\Phi}} := \star_h \frac{\partial_{x_1}\vec{\Phi} \wedge \partial_{x_2}\vec{\Phi}}{|\partial_{x_1}\vec{\Phi} \wedge \partial_{x_2}\vec{\Phi}|} \in \bigwedge\nolimits^{m-2} T_{\vec{\Phi}(x)} M^m,$$

where (x_1, x_2) is a local arbitrary choice of coordinates on $\mathbb{S}^2$ and $\star_h$ is the standard Hodge operator associated to the metric h on multi-vectors in TM.

With these notations we introduce the following definition.

Definition 1.2. A Lipschitz map $\vec{\Phi} \in W^{1,\infty}(\mathbb{S}^2, M^m)$ is called "weak, possibly branched, immersion" if $\vec{\Phi}$ satisfies (1.1) for some $C \geq 1$, if it satisfies (1.2) and if the Gauss map satisfies

$$\int_{\mathbb{S}^2} |D\vec{n}_{\vec{\Phi}}|^2 \, d\mathrm{vol}_g < +\infty, \tag{1.3}$$

where $d\mathrm{vol}_g$ is the volume form associated to $g := \vec{\Phi}^* h$ the pull-back metric of h by $\vec{\Phi}$ on $\mathbb{S}^2$, D denotes the covariant derivative with respect to h and the norm $|D\vec{n}_{\vec{\Phi}}|$ of the tensor $D\vec{n}_{\vec{\Phi}}$ is taken with respect to g on $T^*\mathbb{S}^2$ and h on $\bigwedge^{m-2} TM$. The space of "weak, possibly branched, immersions" of $\mathbb{S}^2$ into M^m is denoted by $\mathcal{F}_{\mathbb{S}^2}$.

For $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$ we denote

$$F(\vec{\Phi}) = \frac{1}{2} \int_{\mathbb{S}^2} |D\vec{n}_{\vec{\Phi}}|^2 \, d\mathrm{vol}_g$$

and

$$G(\vec{\Phi}) := A(\vec{\Phi}) + F(\vec{\Phi}) = \int_{\mathbb{S}^2} \left[1 + \frac{|D\vec{n}_{\vec{\Phi}}|^2}{2} \right] d\mathrm{vol}_g .$$

Using Müller–Svěrák theory of weak isothermic charts (see [18]) and Hélein's moving frame technique (see [8]) one can prove the following (see [20]).

Proposition 1.3. *Let $\vec{\Phi}$ be a weak, possibly branched, immersion of $\mathbb{S}^2$ into M^m in $\mathcal{F}_{\mathbb{S}^2}$. Then there exists a bilipschitz homeomorphism Ψ of $\mathbb{S}^2$ such that $\vec{\Phi} \circ \Psi$ is weakly conformal: it satisfies almost everywhere on $\mathbb{S}^2$*

$$\begin{cases} |\partial_{x_1}(\vec{\Phi} \circ \Psi)|_h^2 = |\partial_{x_2}(\vec{\Phi} \circ \Psi)|_h^2, \\ h(\partial_{x_1}(\vec{\Phi} \circ \Psi), \partial_{x_2}(\vec{\Phi} \circ \Psi)) = 0, \end{cases}$$

where (x_1, x_2) are local arbitrary conformal coordinates in $\mathbb{S}^2$ for the standard metric. Moreover $\vec{\Phi} \circ \Psi$ is in $W^{2,2} \cap W^{1,\infty}(\mathbb{S}^2, M^m)$.

Remark 1.4. In view of Proposition 1.3, a careful reader could wonder why we do not work with conformal $W^{2,2}$ weak, possibly branched, immersions only and why we do not impose for the membership in $\mathcal{F}_{\mathbb{S}^2}$, $\vec{\Phi}$ to be conformal from the beginning. The reason why this would be a wrong strategy and why we have to keep the flexibility for weak immersions not to be necessarily conformal will appear clearly in the second paper [16] while studying the variations of L and while considering general perturbations which do not have to respect infinitesimally the conformal condition.

In the sequel we shall denote by $\mathcal{M}^+(\mathbb{S}^2)$ the non-compact Möbius group of positive conformal diffeomorphisms of the 2-sphere $\mathbb{S}^2$. Our main result in the present work is the following weak-semi-closure result of $\mathcal{F}_{\mathbb{S}^2}$.

Theorem 1.5. *Let $\vec{\Phi}_k$ be a sequence of weak, possibly branched, conformal immersions of $\mathcal{F}_{\mathbb{S}^2}$ such that*

$$\limsup_{k\to+\infty} \int_{\mathbb{S}^2} \left[1 + |D\vec{n}_{\vec{\Phi}_k}|_h^2\right] d\,\mathrm{vol}_{g_k} < +\infty, \tag{1.4}$$

where $d\,\mathrm{vol}_{g_k}$ denotes the volume form associated to the induced metric $g_k := \vec{\Phi}_k^* h$ by $\vec{\Phi}_k$ on $\mathbb{S}^2$ and $D\vec{n}_{\vec{\Phi}_k}$ is the covariant derivative in (M, h) of the normal space $\vec{n}_{\vec{\Phi}_k}$ to $\vec{\Phi}_k$. Assume moreover that

$$\liminf_k \mathrm{diam}(\vec{\Phi}_k(\mathbb{S}^2)) > 0. \tag{1.5}$$

Then there exists a subsequence that we still denote by $\vec{\Phi}_k$, there exists a family of bilipschitz homeomorphisms Ψ_k, there exists a finite family of sequences $(f_k^i)_{i=1,\dots,N}$ of elements in $\mathcal{M}^+(\mathbb{S}^2)$, there exists a finite family of natural integers $(N^i)_{i=1,\dots,N}$ and for each $i \in \{1, \dots, N\}$ there exist finitely many points of $\mathbb{S}^2$, $b^{i,1}, \dots, b^{i,N^i}$ such that

$$\vec{\Phi}_k \circ \Psi_k \to \vec{f}_\infty \quad \textit{strongly in } C^0(\mathbb{S}^2, M^m), \tag{1.6}$$

where $\vec{f}_\infty \in W^{1,\infty}(\mathbb{S}^2, M^m)$, moreover

$$\vec{\Phi}_k \circ f_k^i \rightharpoonup \vec{\xi}_\infty^i \quad \textit{weakly in } W^{2,2}_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{b^{i,1}, \dots, b_{i,N^i}\}), \tag{1.7}$$

where $\vec{\xi}_\infty^j \in \mathcal{F}_{\mathbb{S}^2}$ is conformal. In addition we have

$$\vec{f}_\infty(\mathbb{S}^2) = \bigcup_{i=1}^N \vec{\xi}_\infty^i(\mathbb{S}^2), \tag{1.8}$$

moreover

$$\mathrm{Area}(\vec{\Phi}_k) = \int_{\mathbb{S}^2} 1\, d\,\mathrm{vol}_{g_{\vec{\Phi}_k}} \to \mathrm{Area}(\vec{f}_\infty) = \sum_{i=1}^N \mathrm{Area}(\vec{\xi}_\infty^i), \tag{1.9}$$

and finally

$$(\vec{f}_\infty)_*[\mathbb{S}^2] = \sum_{i=1}^{N} (\vec{\xi}^i_\infty)_*[\mathbb{S}^2], \tag{1.10}$$

where for any Lipschitz mapping $\vec{a}$ from $\mathbb{S}^2$ into M^m, $(\vec{a})_*[\mathbb{S}^2]$ denotes the current given by the push-forward by $\vec{a}$ of the current of integration over $\mathbb{S}^2$: for any smooth two-form ω on M^m

$$\langle (\vec{a})_*[\mathbb{S}^2], \omega \rangle := \int_{\mathbb{S}^2} (\vec{a})^* \omega.$$

In [16], for any given homotopy class γ of $\pi_2(M^m, x_0)$, we will construct such an $\vec{f}_\infty$ representing γ and for which the associated $\vec{\xi}^j_\infty$ are smooth, possibly branched, conformal immersions satisfying the *area constrained Willmore equation*. We have then succeeded in realizing each *based* homotopy class with a smooth mapping of $\mathbb{S}^2$ whose image equals a connected union of smooth *area constrained Willmore*, possibly branched, conformal immersions of $\mathbb{S}^2$.

Before describing the organization of the paper let us recall that the Willmore – and more generally L^2 curvature functionals – have been studied extensively in the Euclidean space (see for instance [2–4, 18, 19, 21, 22, 24]); for some recent result in Riemannian manifolds see [11, 14–17].

The paper is organized as follows: in Section 2 we establish uniform controls of the number of branched points as well as a uniform control of the $L^{2,\infty}$-gradient of the conformal factor. In Section 3 we establish a concentration compactness result for weak branched conformal immersions from an arbitrary Riemann surface into a closed Riemannian manifold. In Section 4 we present a way to "normalize" the parametrization of sequences of conformal weak immersions with uniformly bounded L energy and non-shrinking diameter in order to converges to a non-trivial conformal immersion of $\mathbb{S}^2$. In Section 5 we develop a procedure in order to decompose $\mathbb{S}^2$ into subdomains which, modulo renormalization, will converge to non-trivial conformal immersions of $\mathbb{S}^2$ and which exhaust completely the image. In Section 6 we prove the main Theorem 1.5 and finally in the last section we will prove the extension of the weak closure Theorem 1.5 to bubble trees of elements in $\mathcal{F}_{\mathbb{S}^2}$. This last result will be the starting point to the proof of the realization of homotopy classes by bubble trees of Willmore spheres in [16].

2 Branched points control and $L^{2,\infty}$-estimates of the gradient of the conformal factor in conformal parametrization

Let (M^m, h) be an m-dimensional Riemannian manifold and (Σ, c_o) be a smooth, closed Riemann surface; without loss of generality we can assume that (Σ, c_o)

has a metric with constant curvature and unit area (see for example [10]). First of all we want to define the set of branched conformal immersions with finite total curvature of (Σ, c_0) into (M^m, h). Consider a map $\vec{\Phi} \in W^{1,\infty}(\Sigma, (M, h))$ and consider the following conditions.

(1) Conformality: almost everywhere on Σ

$$\begin{cases} |\partial_{x_1}\vec{\Phi}|_h^2 = |\partial_{x_2}\vec{\Phi}|_h^2, \\ h(\partial_{x_1}\vec{\Phi}, \partial_{x_2}\vec{\Phi}) = 0. \end{cases} \tag{2.1}$$

(2) Finite number of singular points: There exists at most finitely many points $\{b_1, \ldots, b_{N_{\vec{\Phi}}}\}$ such that for any compact $K \subset \mathbb{S}^2 \setminus \{b_1, \ldots, b_{N_{\vec{\Phi}}}\}$

$$\operatorname*{ess\,inf}_{x \in K} |d\vec{\Phi}|(x) > 0. \tag{2.2}$$

Observe that on $\Sigma \setminus \{b_1, \ldots, b_{N_{\vec{\Phi}}}\}$ we can define the normal space

$$\vec{n}_{\vec{\Phi}} \in \Gamma(\vec{\Phi}^{-1}(TM))$$

as

$$\vec{n}_{\vec{\Phi}} := \star_h \frac{\partial_{x_1}\vec{\Phi} \wedge \partial_{x_2}\vec{\Phi}}{|\partial_{x_1}\vec{\Phi} \wedge \partial_{x_2}\vec{\Phi}|}, \tag{2.3}$$

so $\vec{n}$ is defined almost everywhere and is an L^∞ vector field on the whole Σ.

(3) Finite total curvature: we ask that $\vec{n}$ is a $W^{1,2}$ $m-2$-vector field

$$\int_\Sigma |D\vec{n}_{\vec{\Phi}}|^2 dx < \infty. \tag{2.4}$$

Notice that by the invariance of the integrand with respect to conformal changes of metric on the surface Σ, it does not matter which representant in the conformal class we choose to compute (2.4).

A map $\vec{\Phi} \in W^{1,\infty}((\Sigma, c_0), (M, h))$ which satisfies (2.1), (2.2), (2.4) is called *weak branched conformal immersion with finite total curvature* and we denote with

$$\mathcal{F}_\Sigma^{c_0} := \{\vec{\Phi} \in W^{1,\infty}((\Sigma, c_0), (M, h)) \text{ satisfying (2.1), (2.2) and (2.4)}\}. \tag{2.5}$$

2.1 Behaviour at singular points

First of all let us recall the following lemma proved by the second author (see [22, Lemma A.5]. This lemma was originally proved in [18] using quite different arguments. We used instead in [22] the *moving frame approach* of F. Hélein – see [8]). Before stating it let us recall that $\mathrm{Gr}_{p-2}(\mathbb{R}^p)$ denotes the Grassmannian of $p-2$-dimensional linear subspaces of $\mathbb{R}^p$.

Lemma 2.1. *Let $\vec{\Phi}$ be a conformal immersion of $D^2 \setminus \{0\}$ into $\mathbb{R}^p$ in the space $W^{2,2}_{\text{loc}}(D^2 \setminus \{0\}, \mathbb{R}^p)$ and such that the conformal factor* $\log|\nabla\vec{\Phi}| \in L^\infty_{\text{loc}}(D^2 \setminus \{0\})$. *Assume $\vec{\Phi}$ extends to a map in $W^{1,2}(D^2)$ and that the corresponding Gauss map $\vec{n}_{\vec{\Phi}}$ also extends to a map in $W^{1,2}(D^2, \text{Gr}_{p-2}(\mathbb{R}^p))$. Then $\vec{\Phi}$ realizes a Lipschitz conformal immersion of the whole disc D^2 and there exists a positive integer n and a constant C such that*

$$(C - o(1))|z|^{n-1} \leq \left|\frac{\partial\vec{\Phi}}{\partial z}\right| \leq (C + o(1))|z|^{n-1}. \tag{2.6}$$

Observe that if $\vec{\Phi}$ is a weak branched conformal immersion with finite total curvature into the Riemannian manifold (M^m, h), then by the Nash Embedding Theorem we can see the ambient manifold (M^m, h) isometrically embedded in some $\mathbb{R}^n$ and $\vec{\Phi}$ as a weak branched conformal immersion with finite total curvature into $\mathbb{R}^n$ taking values in the submanifold M.

By well-established estimates on the conformal factor (see for example the notes of Rivière [20, pp. 120–136]) we have that

$$\log|\nabla\vec{\Phi}| \in W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}(D^2 \setminus \{0\}),$$

moreover by assumption we have that

$$\vec{n}_{\vec{\Phi}} \in W^{1,2}(D^2);$$

both the information together imply that $\vec{\Phi} \in W^{2,2}_{\text{loc}}(D^2 \setminus \{0\})$. By assumption we have that $\vec{\Phi} \in W^{1,\infty}(D^2)$ so we are in the assumptions of Lemma 2.1 and we can conclude that the singular points $b_1, \ldots, b_{N_{\vec{\Phi}}}$ are actually branch points with *positive* branching order (i.e. $n \geq 2$) given by (2.6).

From the discussion above we know the behaviour of the conformal factor

$$\lambda = \log\left(\left|\frac{\partial\vec{\Phi}}{\partial z}\right|\right)$$

near the branch points $b_1, \ldots, b_{N_{\vec{\Phi}}}$; taking conformal coordinates parametrizing a punctured neighbourhood of b_i on the punctured disc $D^2 \setminus \{0\}$ we observe that the following conditions are satisfied:

$$\begin{cases} -\Delta_0 \lambda = K_{\vec{\Phi}} e^{2\lambda} - K_0 \quad \text{on } \Sigma \setminus \{b_1, \ldots, b_{N_{\vec{\Phi}}}\}, \\ (n_j - 1)\log|z| + (C - o(1)) \leq \lambda(z) \leq (n_j - 1)\log|z| + (C + o(1)), \end{cases} \tag{2.7}$$

where of course $K_{\vec{\Phi}}$ is the Gauss curvature of the metric $g = (\vec{\Phi})^*(h)$ given by the immersion, K_0 is the (constant) Gauss curvature of the reference metric of (Σ, c_0), Δ_0 is the Laplace Beltrami operator on (Σ, c_0) and $n_i \in \mathbb{N}$ are given by Lemma 2.1 applied to a neighbourhood of b_i; observe that the first equation is just the Liouville equation on the regular part of the immersion $\vec{\Phi}$.

Since $\lambda \in L^1_{\mathrm{loc}}(\Sigma)$, we have that $-\Delta_0\lambda$ is a distribution which, using the first equation of (2.7), has singular support contained in $\{b_1, \dots, b_{N_{\vec{\Phi}}}\}$. By Schwarz' lemma it follows that $-\Delta_0\lambda - K_{\vec{\Phi}} e^{2\lambda} + K_0$ is a finite sum of deltas and derivatives of deltas at the points b_i. Using the second condition of (2.7) we get that λ satisfies the following singular PDE on the whole Σ (for more details see [1, Appendix]; see also [13] and [26]):

$$-\Delta_0\lambda = K_{\vec{\Phi}} e^{2\lambda} - K_0 - 2\pi \sum_{j=1}^{N_{\vec{\Phi}}} \big[(n_j - 1)\delta_{b_j}\big], \tag{2.8}$$

where of course δ_{b_j} is the delta centred at the point b_j.

2.2 Estimates on the gradient of the conformal factor and on the sum of the branching orders

Lemma 2.2. *Let $\vec{\Phi} \in \mathcal{F}^{c_0}_{\Sigma}$ be a weak branched conformal immersion of the Riemann surface (Σ, c_0) into the Riemannian manifold (M^m, h). Called $b_1, \dots, b_{N_{\vec{\Phi}}}$ the branch points and $n_1 \geq 1, \dots, n_{N_{\vec{\Phi}}} \geq 1$ the corresponding branching orders given by the previous discussion in Section* 2.1, *we have the following estimates on the sum of the branching orders and on the $L^{2,\infty}$-norm of the conformal factor:*

$$\sum_{j=1}^{N_{\vec{\Phi}}} (n_j - 1) \leq \frac{1}{4\pi} \int_{\Sigma} |D\vec{n}_{\vec{\Phi}}|^2 \, d\mathrm{vol}_g + \frac{1}{2\pi} \Big(\sup_{\vec{\Phi}(\Sigma)} |\bar{K}| \Big) \operatorname{Area}_g(\Sigma) - \chi_E(\Sigma) \tag{2.9}$$

and

$$\|\nabla\lambda\|_{L^{2,\infty}(\Sigma)} \leq C_{(\Sigma,c_0)} \left[1 + \int_{\Sigma} |D\vec{n}_{\vec{\Phi}}|^2 \, d\mathrm{vol}_g + \Big(\sup_{\vec{\Phi}(\Sigma)} |\bar{K}| \Big) \operatorname{Area}_g(\Sigma) \right], \tag{2.10}$$

where Area_g *is the area of Σ with respect to the metric $g := \vec{\Phi}^* h$, $\bar{K}$ is the sectional curvature of (M^m, h), $\chi_E(\Sigma)$ is the Euler characteristic of Σ, $C_{(\Sigma,c_0)}$ is a constant depending only on the Riemann surface (Σ, c_0), and the $L^{2,\infty}$-norm on Σ is taken with respect to the metric of scalar constant curvature and volume* 1 *on (Σ, c_0).*

Proof. By the Gauss equation (which still holds almost everywhere on a weak branched conformal immersion with finite total curvature by an approximation argument, in the smooth case see for example [5, p. 130]) we have

$$K_{\vec{\Phi}} = \bar{K}(\vec{\Phi}_*(T\Sigma)) + \langle \mathbb{I}_{11}, \mathbb{I}_{22} \rangle - |\mathbb{I}_{12}|^2, \tag{2.11}$$

where $\bar{K}(\vec{\Phi}_*(T\Sigma))$ is the sectional curvature of (M,h) computed on the plane $\vec{\Phi}_*(T\Sigma)$, $K_{\vec{\Phi}}$ is the Gauss curvature of (Σ, g) and $\mathbb{I}_{ij}$ is the second fundamental form which is defined almost everywhere in the usual sense:

$$\mathbb{I}_{ij} := -\sum_{\alpha=1}^{m-2} \left\langle D_{\vec{e}_i}\vec{n}_\alpha, \vec{e}_j \right\rangle \vec{n}_\alpha,$$

where

$$\vec{e}_i(q) := \frac{1}{|\partial_{x_i}\vec{\Phi}|}\partial_{x_i}\vec{\Phi}|_q$$

is an orthonormal frame of $\vec{\Phi}_*(T_q\Sigma)$ and $(\vec{n}_1(q),\dots,\vec{n}_{m-2}(q))$ realizes a positive orthonormal basis of $(\vec{\Phi}_*(T_q\Sigma))^\perp$ so that

$$\vec{n}_{\vec{\Phi}} = \vec{n}_1 \wedge \cdots \wedge \vec{n}_{m-2}.$$

Since $|\mathbb{I}|^2 = |D\vec{n}_{\vec{\Phi}}|^2$, by Schwarz's inequality we have the estimate

$$\begin{aligned} |K_{\vec{\Phi}}| &\le |\bar{K}(\vec{\Phi}_*(T\Sigma))| + |\mathbb{I}_{11}||\mathbb{I}_{22}| + |\mathbb{I}_{12}|^2 \\ &\le |\bar{K}(\vec{\Phi}_*(T\Sigma))| + \frac{1}{2}\left(|\mathbb{I}_{11}|^2 + |\mathbb{I}_{22}|^2 + 2|\mathbb{I}_{12}|^2\right) \\ &= |\bar{K}(\vec{\Phi}_*(T\Sigma))| + \frac{1}{2}|D\vec{n}_{\vec{\Phi}}|^2. \end{aligned} \tag{2.12}$$

Integrating equation (2.8) and using the estimate (2.12) we get the following estimate on the number of branch points in terms of the total curvature and the area:

$$\begin{aligned} 2\pi\sum_{j=1}^{N_{\vec{\Phi}}}(n_j-1) &= \int_\Sigma (K_{\vec{\Phi}}e^{2\lambda})\, d\mathrm{vol}_{c_0} - K_0 \\ &\le \frac{1}{2}\int_\Sigma |D\vec{n}_{\vec{\Phi}}|^2\, d\mathrm{vol}_g + \Big(\sup_{\vec{\Phi}(\Sigma)} |\bar{K}|\Big)\,\mathrm{Area}_g(\Sigma) \\ &\qquad - 2\pi\chi_E(\Sigma), \end{aligned} \tag{2.13}$$

where $d\mathrm{vol}_{c_0}$ is the volume form with respect to constant scalar curvature metric of volume 1 on (Σ, c_0). Now putting together equation (2.8) and estimate (2.13) we get the following bound on the norm of $-\Delta_0\lambda$ as Radon measure:

$$\|\Delta_0\lambda\|_{\mathcal{M}(\Sigma)} \le 4\pi|\chi_E(\Sigma)| + \int_\Sigma |D\vec{n}_{\vec{\Phi}}|^2\, d\mathrm{vol}_g + 2\Big(\sup_{\vec{\Phi}(\Sigma)} |\bar{K}|\Big)\,\mathrm{Area}_g(\Sigma) \tag{2.14}$$

which implies that there exists a constant $C = C_{(\Sigma,c_0)}$ depending just on the Riemann surface (Σ, c_0) such that (see [8, p. 138])

$$\|\nabla\lambda\|_{L^{2,\infty}(\Sigma)} \le C_{(\Sigma,c_0)}\left[1+\int_\Sigma |D\vec{n}_{\vec{\Phi}}|^2\, d\mathrm{vol}_g + \Big(\sup_{\vec{\Phi}(\Sigma)} |\bar{K}|\Big)\,\mathrm{Area}_g(\Sigma)\right] \tag{2.15}$$

which concludes the proof. □

3 Concentration compactness for weak branched conformal immersions of general Riemann surfaces into manifolds

Proposition 3.1. *Let (Σ, c_k) be a sequence of Riemann surfaces defined on the same topological surface Σ, consider weak branched conformal immersions*

$$\mathcal{F}_\Sigma^{c_k} \ni \vec{\Phi}_k \hookrightarrow (M^m, h)$$

into the m-dimensional Riemannian manifold (M^m, h) and assume the following conditions hold true:

(i) *the conformal structures c_k are contained in a fixed compact subset of the moduli space of Σ,*

(ii) *the areas of Σ induced by the immersions are uniformly bounded from above: called $g_k := \vec{\Phi}_k^*(h)$,*

$$\mathrm{Area}_{g_k}(\Sigma) \le C,$$

(iii) *the energies of the $\vec{\Phi}_k$ are uniformly bounded from above:*

$$F(\vec{\Phi}_k) = \frac{1}{2}\int_\Sigma |D\vec{n}_{\vec{\Phi}_k}|^2\, d\mathrm{vol}_{g_k} = \frac{1}{2}\int_\Sigma |\mathbb{I}_{\vec{\Phi}_k}|^2\, d\mathrm{vol}_{g_k} \le C.$$

Then there exists a finite set of points $\{a_1, \ldots, a_N\} \subset \Sigma$ such that, called

$$\lambda_k := \log|\partial_{x_1}\vec{\Phi}_k|_h = \log|\partial_{x_2}\vec{\Phi}_k|_h$$

the conformal factor of the immersion $\vec{\Phi}_k$, for any compact $K \Subset \Sigma \setminus \{a_1, \ldots, a_N\}$ the following properties hold:

(a) *there exists a constant C_K depending on K and on the bounds on areas and energies of $\vec{\Phi}_k$ such that, up to subsequences on k,*

$$\|e^{\lambda_k}\|_{L^\infty(K)} \le C_K,$$

(b) *either there exists a constant C_K depending as above such that, up to subsequences on k,*

$$\|\lambda_k\|_{L^\infty(K)} \le C_K$$

or, up to subsequences on k,

$$\lambda_k \to -\infty \quad \textit{uniformly on } K.$$

Proof. Consider a Nash isometric embedding $\vec{I} : (M^m, h) \hookrightarrow \mathbb{R}^n$; in this way $\vec{I} \circ \vec{\Phi}_k : (\Sigma, c_k) \hookrightarrow (M^m, h) \hookrightarrow \mathbb{R}^n$ are weak branched conformal immersions in $\mathbb{R}^n$. Observe that by the compactness of M and the area bound on $\vec{\Phi}_k$, the energies of $\vec{I} \circ \vec{\Phi}_k$ are also uniformly bounded:

$$\int_\Sigma |d\vec{n}_{\vec{I}\circ\vec{\Phi}_k}|^2_{g_k}\, d\mathrm{vol}_{g_k} \le C. \tag{3.1}$$

Indeed, we have

$$|d\vec{n}_{\vec{I}\circ\vec{\Phi}_k}|^2_{g_k} \le |D^h \vec{n}_{\vec{\Phi}_k}|^2_{g_k} + |d\vec{n}_{\vec{I}}|^2_{g_k}, \tag{3.2}$$

where $\vec{n}_{\vec{I}}$ is the Gauss map of M^m in $\mathbb{R}^n$. This inequality comes simply from the fact that

$$\vec{n}_{\vec{I}\circ\vec{\Phi}_k} = \vec{n}_{\vec{\Phi}_k} \wedge \vec{n}_{\vec{I}}.$$

In order to get the claim it suffices to integrate on Σ with respect to $d\mathrm{vol}_{g_k}$ and recall the assumed bounds on area and energy of $\vec{\Phi}_k$. Now, since the integrand in (3.1) is invariant under conformal change of metric in the domain, we have

$$\int_\Sigma |d\vec{n}_{\vec{I}\circ\vec{\Phi}_k}|^2_{h_k}\, d\mathrm{vol}_{h_k} = \int_\Sigma |d\vec{n}_{\vec{I}\circ\vec{\Phi}_k}|^2_{g_k}\, d\mathrm{vol}_{g_k} \le C, \tag{3.3}$$

where h_k is the smooth reference metric on (Σ, c_k) of constant curvature and unit volume. Since c_k by assumption are contained in a compact subset of the moduli space of Σ, up to subsequences, h_k strongly converges in $C^r(\Sigma)$ for every r to h_∞, the constant curvature metric of volume 1 associated to c_∞ (the limiting conformal structure). Recall that $\vec{\Phi}_k$ are weak branched conformal immersions with branch points $\{b_k^1, \dots, b_k^{N_k}\}$; from the area and energy bounds, Lemma 2.2 implies that the number of branch points N_k is uniformly bounded, so up to subsequences we can assume N_k constant, say $N_k = N_1$. For each $k \in \mathbb{N}$ and $x \in \Sigma$ we assign $\rho_x^k > 0$ such that

$$\int_{B_{\rho_x^k}(x)} |d\vec{n}_{\vec{I}\circ\vec{\Phi}_k}|^2_{h_k}\, d\mathrm{vol}_{h_k} = \frac{8\pi}{3}, \tag{3.4}$$

where $B_{\rho_x^k}(x)$ is the geodesic ball in (Σ, h_k) of centre x and radius ρ_x^k (notice that for any closed surface Σ immersed in $\mathbb{R}^n$ one has $\int_\Sigma |d\vec{n}|^2\, d\mathrm{vol}_g \ge 8\pi$ so the minimal radius ρ_x^k exists finite). From the covering $\{B_{\rho_x^k}(x)\}_{x\in\Sigma}$ we extract a finite Besicovich covering: every point of the surface Σ is covered by at most $\xi = \xi(\Sigma, h_\infty) \in \mathbb{N}$ balls. Let $\{B_{\rho_k^i}(x_k^i)\}_{i\in I}$ be such a cover and observe that up to subsequences: I is independent on k, $x_k^i \to x_\infty^i$, $\rho_k^i \to \rho_\infty^i$, $b_k^i \to b_\infty^i$. Let

$$J := \{i \in I : \rho_\infty^i = 0\} \quad \text{and} \quad I_0 := I \setminus J.$$

The union of the closed balls $\bigcup_{i\in I_0} \bar{B}_{\rho^i_\infty}(x^i_\infty)$ covers Σ. Because of the strict convexity of the balls with respect to the Euclidean distance ($\Sigma =$ Torus, or $\Sigma = \mathbb{S}^2$ via stereographic projection) or the hyperbolic distance (genus$(\Sigma) > 1$) the points of Σ which are not contained in the union of the *open* balls $\bigcup_{i\in I} B_{\rho^i_\infty}(x^i_\infty)$ cannot accumulate and therefore are isolated and hence finite (this argument was the same in the unbranched situation, see [22]). Denote

$$\{d^1, \ldots, d^{N_2}\} := \Sigma \setminus \bigcup_{i\in I_0} B_{\rho^i_\infty}(x^i_\infty).$$

Now let

$$\{a^1, \ldots, a^N\} := \{d^1, \ldots, d^{N_2}\} \cup \{b^1_\infty, \ldots, b^{N_1}_\infty\}$$

and fix a compact subset $K \Subset \Sigma\setminus\{a^1, \ldots, a^N\}$. Clearly there exists a $\delta > 0$ such that $K \subset \Sigma \setminus \bigcup_{i=1}^N \bar{B}_\delta(a^i)$. Since $\Sigma \setminus \bigcup_{i=1}^N B_\delta(a^i) \subset \bigcup_{i\in I_0} B_{\rho^i_\infty}(x^i_\infty)$, there exist $0 < r^i < \rho^i_\infty$ such that

$$\Sigma \setminus \bigcup_{i=1}^N B_\delta(a^i) \subset \bigcup_{i\in I_0} B_{r^i}(x^i_\infty) \tag{3.5}$$

and for k large enough one has, for any $i \in I_0$,

$$B_{r^i}(x^i_\infty) \subset B_{\rho^i_k}(x^i_k).$$

Let $s^i = (r^i + \rho^i_\infty)/2$. Again, for k large enough, $B_{s^i}(x^i_\infty) \subset B_{\rho^i_k}(x^i_k)$ for $i \in I_0$. Recall that by construction we have the crucial estimate

$$\int_{B_{s^i}(x^i_\infty)} |d\vec{n}_{\vec{I}\circ\vec{\Phi}_k}|^2_{h_k}\, d\mathrm{vol}_{h_k} \le \frac{8\pi}{3}. \tag{3.6}$$

Now we claim that for every i and, up to subsequences, k there exists a constant $\bar{\lambda}^i_k$ such that for every radius $r^i < r < s^i$:

$$\|\lambda_k - \bar{\lambda}^i_k\|_{B_r(x^i_\infty)\setminus\bigcup_{j=1}^{N_1} B_{\delta/2}(b^j_\infty)} \le C_{r,\delta}. \tag{3.7}$$

In order to prove the claim observe that for each ball $B_{s^i}(x^i_\infty)$, $i \in I_0$, we have two possibilities: either it contains a limit of branch points $b^{j_i}_\infty$ or not. In the second case, since (3.6) holds, we can apply the construction of moving frames ([22, pp. 23, 49], [20, pp. 139–142]) and on the slightly smaller ball $B_r(x^i_\infty)$ we have an L^∞ control of the conformal factor λ_k up to a constant $\bar{\lambda}^i_k$ as desired. In the first case $B_{s^i}(x^i_\infty)$ contains some limit of branch points. Consider a finite cover of

$$B_{s^i}(x^i_\infty) \setminus \bigcup_{j=1}^{N_1} B_{\delta/4}(b^j_\infty)$$

of balls contained in $B_{\rho^i_k}(x^i_k) \setminus \bigcup_{j=1}^{N_1} B_{\delta/8}(b^j_\infty)$ (recall that, for large k, we have that $B_{s^i}(x^i_\infty) \subset B_{\rho^i_k}(x^i_k)$ with boundaries at strictly positive distance); so for each ball $\tilde{B}$ in this last covering we have

$$\int_{\tilde{B}} |d\vec{n}_{\vec{I}\circ\vec{\Phi}_k}|^2_{h_k}\, d\mathrm{vol}_{h_k} \leq \frac{8\pi}{3}$$

for large k. Hence on each ball in the covering we construct Hélein's moving frames as before getting L^∞ control of λ_k on any slightly smaller ball. This gives the estimate (3.7) on

$$B_r(x^i_\infty) \setminus \bigcup_{j=1}^{N_1} B_{\delta/2}(b^j_\infty) \subset B_{s^i}(x^i_\infty) \setminus \bigcup_{j=1}^{N_1} B_{\delta/4}(b^j_\infty).$$

Now, from the area bound, $\bar{\lambda}^i_k \leq C$ independent on k and i. Indeed if it is not the case, there would exist i_0 and a subsequence in k such that $\bar{\lambda}^{i_0}_k \to +\infty$; therefore by (3.7)

$$\begin{aligned}
&\mathrm{Area}_{g_k}\left(B_r(x^{i_0}_\infty) \setminus \bigcup_{j=1}^{N_1} B_{\delta/2}(b^j_\infty)\right)\\
&\quad = \int_{B_r(x^{i_0}_\infty)\setminus\bigcup_{j=1}^{N_1} B_{\delta/2}(b^j_\infty)} e^{2\lambda_k}\, d\mathrm{vol}_{h_k}\\
&\quad \geq e^{-2C_{r,\delta}} \int_{B_r(x^{i_0}_\infty)\setminus\bigcup_{j=1}^{N_1} B_{\delta/2}(b^j_\infty)} e^{2\bar{\lambda}^{i_0}_k}\, d\mathrm{vol}_{h_k}
\end{aligned}$$

contradicting the area bound, so (a) is proved.

For getting (b) observe that if for one index i_0 there exists a constant such that $\bar{\lambda}^{i_0}_k \geq C > -\infty$ for infinitely many k, then by (3.7) all the balls of the covering with non-empty intersection with such a ball have the same property (the holes $\bigcup_{j=1}^{N_1} B_{\delta/2}(b^j_\infty)$ do not cover the intersection of two balls of the covering if $\delta > 0$ is taken small enough); then, since Σ is connected, on all the balls of the covering we have $\bar{\lambda}^i_k \geq C$ and λ_k are uniformly bounded on the compact subset K; on the other hand, if on every ball of the covering $\bar{\lambda}^i_k \to -\infty$ (up to subsequences in k), we have by (3.7) that $\lambda_k \to -\infty$ uniformly on the compact subset K. Therefore also (b) is proved. □

4 A "good gauge extraction" in the Möbius group for sequences of immersions of spheres under G-energy control and diameter positive lower bound

In the present section we prove that, assuming the reference surface is $\mathbb{S}^2$ and that the images $\Phi_k(\mathbb{S}^2)$ have diameter bounded below by a strictly positive constant, we can reparametrize the immersions such that in Proposition 3.1 the degenerating case $\lambda_k \to -\infty$ does not occur; therefore on compact subsets invading $\mathbb{S}^2$ there is an L^∞ control of the conformal factors.

Lemma 4.1 (Good Gauge Extraction Lemma). *Let $\mathcal{F}_{\mathbb{S}^2} \ni \vec{\Phi}_k \hookrightarrow (M^m, h)$ be a sequence of weak branched conformal immersions of $\mathbb{S}^2$ into the m-dimensional Riemannian manifold (M^m, h) and assume the following conditions hold true:*

(i) *the areas of $\mathbb{S}^2$ induced by the immersions are uniformly bounded from above: called $g_k := \vec{\Phi}_k^*(h)$,*

$$\mathrm{Area}_{g_k}(\mathbb{S}^2) \leq C,$$

(ii) *the energies of the $\vec{\Phi}_k$ are uniformly bounded from above:*

$$F(\vec{\Phi}_k) = \frac{1}{2}\int_{\mathbb{S}^2} |D\vec{n}_{\vec{\Phi}_k}|^2 \, d\mathrm{vol}_{g_k} = \frac{1}{2}\int_{\mathbb{S}^2} |\mathbb{I}_{\vec{\Phi}_k}|^2 \, d\mathrm{vol}_{g_k} \leq C,$$

(iii) *the diameters of $\vec{\Phi}_k(\mathbb{S}^2)$ are bounded below by a strictly positive constant:*

$$\mathrm{diam}_M(\vec{\Phi}_k(\mathbb{S}^2)) \geq \frac{1}{C}.$$

Then for every k (up to subsequences) there exists a positive Möbius transformations $f_k \in \mathcal{M}^+(\mathbb{S}^2)$ such that, called

$$\mathcal{F}_{\mathbb{S}^2} \ni \tilde{\vec{\Phi}}_k := \vec{\Phi}_k \circ f_k$$

the reparametrized immersion and

$$\tilde{\lambda}_k := \log |\partial_{x_1} \tilde{\vec{\Phi}}_k| = \log |\partial_{x_2} \tilde{\vec{\Phi}}_k|$$

the new conformal factor, the following holds: There exists a finite set of points $\{a_1, \dots, a_N\} \subset \mathbb{S}^2$ such that for any compact subset $K \Subset \mathbb{S}^2 \setminus \{a_1, \dots, a_N\}$ there exists a constant C_K depending on the compact K and on the bounds on areas and energies of $\vec{\Phi}_k$ such that, up to subsequences on k,

$$\|\tilde{\lambda}_k\|_{L^\infty(K)} \leq C_K.$$

Proof. We prove the lemma by contradiction. If the thesis is not true, then for every $f_k : \mathbb{S}^2 \to \mathbb{S}^2$ composition of isometries and dilations, called

$$\tilde{\vec{\Phi}}_k := \vec{\Phi}_k \circ f_k$$

the reparametrized immersion and

$$\tilde{\lambda}_k := \log |\partial_{x_1} \tilde{\vec{\Phi}}_k| = \log |\partial_{x_2} \tilde{\vec{\Phi}}_k|$$

the new conformal factor, the second case in (b) of Proposition 3.1 occurs: there exists a finite set of points $\{a_1, \dots, a_N\} \subset \mathbb{S}^2$ such that for any compact subset $K \Subset \mathbb{S}^2 \setminus \{a_1, \dots, a_N\}$, up to subsequences on k,

$$\tilde{\lambda}_k \to -\infty \quad \text{uniformly on } K.$$

On the other hand, called $\vec{I} : M \to \mathbb{R}^p$ the Nash isometric embedding, combining assumptions (i) and (ii) as in the beginning of the proof of Proposition 3.1 we have (3.1), namely

$$\int_{\mathbb{S}^2} |d\vec{n}_{\vec{I}\circ\vec{\Phi}_k}|^2_{g_k}\, d\mathrm{vol}_{g_k} \le C. \tag{4.1}$$

We will prove that if the thesis is not true and assumption (iii) holds, then (4.1) cannot be satisfied.

Fix any $2 \le n \in \mathbb{N}$, take $p^1_k, \dots, p^n_k \in \vec{\Phi}_k(\mathbb{S}^2)$ such that

$$\mathrm{dist}_M(p^i_k, p^j_k) \ge \frac{1}{Cn} \quad \text{for all } i \ne j \tag{4.2}$$

(clearly this is can be done thanks to assumption (iii)) and pick $q^i_k \in \mathbb{S}^2$ such that $\vec{\Phi}_k(q^i_k) = p^i_k$ (observe that the order in the i indicization is not relevant; in the sequel we will relabel the points for convenience of notation). Considering stereographic projection with centre different from all the points $\{q^i_k\}_{i,k}$ we can carry on the proof in $\mathbb{R}^2$.

Step 1: for every k consider the two points at minimal distance; up to subsequences in k and relabeling in i we can assume they are q^1_k and q^2_k:

$$|q^1_k - q^2_k|_{\mathbb{R}^2} \le |q^i_k - q^j_k|_{\mathbb{R}^2} \quad \text{for all } k \text{ and all } i \ne j.$$

Up to compositions $f_k^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ of isometries and dilations, we can assume that $q^1_k = (0,0)$ is the origin and $q^2_k = (1,0)$. By construction, for fixed k, all the points q^i_k are at mutual distance at least 1 and up to subsequences, either $|q^1_k| \to \infty$ as $k \to \infty$ or

$$q^i_k \to q^i_\infty \quad \text{with} \quad |q^i_\infty - q^j_\infty| \ge 1 \quad \text{for all } i \ne j.$$

Since we are assuming the thesis is not true, as explained above, we have that there exists a finite set of points $\{a_1, \dots, a_N\} \subset \mathbb{S}^2$ such that for any compact subset $K \Subset \mathbb{S}^2 \setminus \{a_1, \dots, a_N\}$, up to subsequences on k,

$$\lambda_k \to -\infty \quad \text{uniformly on } K \tag{4.3}$$

(with abuse of notation we identify $\vec{\Phi}_k$ with $\tilde{\vec{\Phi}}_k$ and λ_k with $\tilde{\lambda}_k$).

It is clear that one can construct two smooth open subsets

$$C^1 \subset B_{\frac{2}{3}}((0,0)) \quad \text{and} \quad C^2 \subset B_{\frac{2}{3}}((1,0))$$

such that

- $\bar{C}^1$ and $\bar{C}^2$ are diffeomorphic to closed balls,
- the intersection of $\bar{C}^1$ and $\bar{C}^2$ consists of just one point, say q_0:
$$\bar{C}^1 \cap \bar{C}^2 = \{q_0\} = \partial C^1 \cap \partial C^2, \tag{4.4}$$
- $q_k^1 = (0,0) \in C^1$ and $q_k^2 = (1,0) \in C^2$ but observe that by construction
$$\mathrm{dist}_{\mathbb{R}^2}(q_k^i, C^1 \cup C^2) \geq \frac{1}{3} \quad \text{for every } i \geq 3,$$
- $(\partial C^1 \cup \partial C^2) \cap \{a_1, \dots, a_N\} = \emptyset$.

Observe that the last condition together with (4.3) implies that the lengths of the images of ∂C^1 and ∂C^2 converge to 0:

$$\mathcal{H}^1\big(\vec{\Phi}_k(\partial C^1) \cup \vec{\Phi}_k(\partial C^2)\big) \to 0. \tag{4.5}$$

Recall that the branch points $\{b_k^1, \dots, b_k^{N_k}\}$ of Φ_k converge up to subsequences to a subset of $\{a_1, \dots, a_N\}$, then we also have (this information will be used later in Step 2)

$$\big(\partial C^1 \cup \partial C^2\big) \cap \{b_k^1, \dots, b_k^{N_k}\} = \emptyset \quad \text{for large } k. \tag{4.6}$$

Notice moreover that at least one between $\vec{\Phi}_k(C^1)$ and $\vec{\Phi}_k(C^2)$ has diameter at least $\frac{1}{2nC}$, say $\vec{\Phi}_k(C^1)$:

$$\mathrm{diam}_M(\vec{\Phi}_k(C^1)) \geq \frac{1}{2nC}. \tag{4.7}$$

Indeed if

$$\mathrm{diam}_M(\vec{\Phi}_k(C^1)) < \frac{1}{2nC} \quad \text{and} \quad \mathrm{diam}_M(\vec{\Phi}_k(C^2)) < \frac{1}{2nC},$$

then since $\vec{\Phi}_k(C^1)$ and $\vec{\Phi}_k(C^2)$ are connected throughout $\vec{\Phi}_k(q_0)$,

$$\mathrm{dist}_M(p_k^1, p_k^2) = \mathrm{dist}_M(\vec{\Phi}_k(q_k^1), \vec{\Phi}_k(q_k^2)) < \frac{1}{nC}$$

contradicting (4.2).

Now recall [21, Lemma I.1] for weak immersions without branch points: let Σ be a compact surface with boundary and let $\vec{\Phi} : \Sigma \hookrightarrow \mathbb{R}^p$ be a weak immersion in $\mathbb{R}^p$ without branch points; then the following inequality holds:

$$4\pi \leq \int_\Sigma |\vec{H}|^2 \, d\mu_g + 2\frac{\mathcal{H}^1(\vec{\Phi}(\partial\Sigma))}{d(\vec{\Phi}(\partial\Sigma), \vec{\Phi}(\Sigma))}, \tag{4.8}$$

where d is the usual distance between two sets:

$$d(\vec{\Phi}(\partial\Sigma), \vec{\Phi}(\Sigma)) := \sup_{p_1 \in \vec{\Phi}(\Sigma)} \inf_{p_2 \in \vec{\Phi}(\partial\Sigma)} |p_1 - p_2|_{\mathbb{R}^p}.$$

Let $b_k^1, \dots, b_k^{L_k}$ be the branch points of $\vec{\Phi}_k$ contained in C^1; then apply (4.8) to $\vec{\Phi}_k$ restricted to $\bar{C}^1 \setminus (\bigcup_{j=1}^{L_k} B_\epsilon(b_k^j))$ (by construction b_k^j converges to some $a_j \notin \partial C^1$ as $k \to \infty$ so for large k and small ϵ, $B_\epsilon(b_k^j) \subset C^1$ and the difference set above is smooth; clearly this restriction of $\vec{\Phi}_k$ is a weak immersion without branch points); as $\epsilon \to 0$ we obtain

$$4\pi \leq \int_{C^1} |\vec{H}_{\vec{I}} \circ \vec{\Phi}_k|^2 \, d\mathrm{vol}_{g_k} + 2\frac{\mathcal{H}^1(\vec{\Phi}_k(\partial C^1))}{d(\vec{\Phi}_k(\partial C^1) \cup (\bigcup_{j=1}^{L_k} \vec{\Phi}_k(b_k^j)), \vec{\Phi}_k(C^1))}. \tag{4.9}$$

Observe that

$$\liminf_{k\to\infty} d\left(\vec{\Phi}_k(\partial C^1) \bigcup \left(\bigcup_{j=1}^{L_k} \vec{\Phi}_k(b_k^j)\right), \vec{\Phi}_k(C^1)\right) > 0. \tag{4.10}$$

Indeed, if up to subsequences

$$d\left(\vec{\Phi}_k(\partial C^1) \bigcup \left(\bigcup_{j=1}^{L_k} \vec{\Phi}_k(b_k^j)\right), \vec{\Phi}_k(C^1)\right) \to 0,$$

then, since by (4.5) we have $\mathcal{H}^1(\vec{\Phi}_k(\partial C^1)) \to 0$, it is easy to see that we would have also $\mathrm{diam}_M(\vec{\Phi}_k(C^1)) \to 0$; contradicting (4.7).

Now, using (4.10) and (4.5), formula (4.9) gives

$$4\pi - \frac{1}{n-1} \leq \int_{C^1} |\vec{H}_{\vec{I}} \circ \vec{\Phi}_k|^2 \, d\mathrm{vol}_{g_k} \quad \text{for large } k. \tag{4.11}$$

In other words, for large k we isolated a region $C^1 \subset \mathbb{R}^2$ containing q_k^1, with distance at least $\frac{1}{3}$ from all the other points q_k^i, with Willmore energy at least $4\pi - \frac{1}{n-1}$.

Step 2: we continue the proof by induction.

Base of the induction. Let $2 \leq i_0 \leq n-1$ and assume the following properties are satisfied:

(1) for every $i < i_0$ and every k up to subsequences there exists a smooth open subset $C_k^i \subset B_{r_i}(q_k^i) \subset \mathbb{R}^2$, $r_i > 0$, independent on k, such that

   (a) $q_k^i \in C_k^i$ but $q_k^j \notin \bar{C}_k^i$ for every $i, j < i_0, i \neq j$,

   (b) $\mathcal{H}^1(\vec{\Phi}_k(\partial C_k^i)) \to 0$ as $k \to \infty$,

   (c) $\partial C_k^i \cap \{b_k^1, \dots, b_k^{N_k}\} = \emptyset$ where, as before, $b_k^1, \dots, b_k^{N_k}$ are the branch points of the immersion $\vec{\Phi}_k$,

(d) $4\pi - \frac{1}{n-1} \leq \int_{C_k^i} |\vec{H}_{\vec{I}} \circ \vec{\Phi}_k|^2 \, d\mathrm{vol}_{g_k}$,
(e) $C_k^i \cap C_k^j = \emptyset$ for every $i, j < i_0, i \neq j$,
(f) if there exists an $i_1 < i_0$ such that for some $j \geq i_0$ it happens that

$$\mathrm{dist}_{\mathbb{R}^2}(q_k^j, C_k^{i_1}) \to 0 \quad \text{as } k \to \infty$$

up to subsequences, then for every $i \leq i_1$ we have

$$\mathrm{diam}_{\mathbb{R}^2}(C_k^i) \to 0 \quad \text{as } k \to \infty$$

on the same subsequence,

(2) $|q_k^i - q_k^j|_{\mathbb{R}^2} \geq 1$ for every $i, j \geq i_0$.

Inductive step. For every k there exists a smooth open subset $C_k^{i_0} \subset \mathbb{R}^2$ and there exists $f_k : \mathbb{R}^2 \to \mathbb{R}^2$ composition of dilations and isometries such that called $\tilde{q}_k^i := f_k^{-1}(q_k^i)$ and $\tilde{C}_k^i := f_k^{-1}(C_k^i)$ the following properties hold:

(1) (a) up to relabeling in j the points $\tilde{q}_k^j$, $j \geq i_0$, and up to subsequences in k we have $\tilde{q}_k^{i_0} \in C_k^{i_0}$ but $\tilde{q}_k^j \notin \bar{C}_k^{i_0}$ for every $j \neq i_0$, moreover

$$C_k^{i_0} \subset B_{r_{i_0}}(\tilde{q}_k^{i_0}) \subset \mathbb{R}^2$$

with r_{i_0} independent of k,

(b) $\mathcal{H}^1(\vec{\Phi}_k(\partial C_k^{i_0})) \to 0$ as $k \to \infty$,

(c) $\partial C_k^{i_0} \cap \{\tilde{b}_k^1, \ldots, \tilde{b}_k^{N_k}\} = \emptyset$ where $\tilde{b}_k^1 := f_k^{-1}(b_k^1), \ldots, \tilde{b}_k^{N_k} := f_k^{-1}(b_k^{N_k})$ are the branch points of

$$\tilde{\vec{\Phi}}_k := \vec{\Phi}_k \circ f_k,$$

(d) $4\pi - \frac{1}{n-1} \leq \int_{C_k^{i_0}} |\vec{H}_{\vec{I}} \circ \vec{\Phi}_k|^2 \, d\mathrm{vol}_{g_k}$,
(e) $C_k^{i_0} \cap \tilde{C}_k^j = \emptyset$ for every $j < i_0$,
(f) $\liminf_k \mathrm{dist}_{\mathbb{R}^2}(C_k^{i_0}, \tilde{q}_k^j) > 0$ for every $j > i_0$,

(2) $|\tilde{q}_k^i - \tilde{q}_k^j|_{\mathbb{R}^2} \geq 1$ for every $i, j \geq i_0 + 1$.

Let us prove the inductive step assuming the base of the induction is satisfied.

As in Step 1 above for every k consider the two points at minimal distance among $\{q_k^i\}_{i \geq i_0}$; up to subsequences in k and relabeling in i we can assume these two points are $q_k^{i_0}$ and $q_k^{i_0+1}$:

$$|q_k^{i_0} - q_k^{i_0+1}|_{\mathbb{R}^2} \leq |q_k^i - q_k^j|_{\mathbb{R}^2} \quad \text{for all } k \text{ and all } i, j \geq i_0, i \neq j.$$

Consider $f_k : \mathbb{R}^2 \to \mathbb{R}^2$ composition of isometries and dilations such that

$$\tilde{q}_k^{i_0} := f_k^{-1}(q_k^{i_0}) = (0, 0)$$

is the origin and

$$\tilde{q}_k^{i_0+1} := f_k^{-1}(q_k^{i_0+1}) = (1,0).$$

By construction, for fixed k, all the points $\{\tilde{q}_k^i\}_{i\geq i_0}$ are at mutual distance at least 1 and up to subsequences either $|\tilde{q}_k^i| \to \infty$ as $k \to \infty$ or

$$\tilde{q}_k^i \to \tilde{q}_\infty^i \quad \text{with} \quad |\tilde{q}_\infty^i - \tilde{q}_\infty^j| \geq 1 \quad \text{for all } i, j \geq i_0, i \neq j. \tag{4.12}$$

Since we are assuming the thesis is not true, as in Step 1, we have that there exists a finite set of points $\{a_1, \dots, a_N\} \subset \mathbb{S}^2$ such that for any compact subset $K \Subset \mathbb{S}^2 \setminus \{a_1, \dots, a_N\}$, up to subsequences on k,

$$\tilde{\lambda}_k \to -\infty \quad \text{uniformly on } K, \tag{4.13}$$

where $\tilde{\lambda}_k = \log|\partial_{x_1}\tilde{\vec{\Phi}}_k| = \log|\partial_{x_2}\tilde{\vec{\Phi}}_k|$ is the conformal factor of $\tilde{\vec{\Phi}}_k := \vec{\Phi}_k \circ f_k$. Observe that if

$$\limsup_{k\to\infty} |q_k^{i_0} - q_k^{i_0+1}| < +\infty,$$

then the situation described in (1f) of the base of induction is exactly the same for the rescaled quantities $\tilde{q}_k^j, \tilde{C}_k^i$. On the contrary, if for some subsequence in k we have $|q_k^{i_0} - q_k^{i_0+1}| \to +\infty$, then we are rescaling with a diverging ratio and, on the same subsequence, $\lim_{k\to\infty} \operatorname{diam}_{\mathbb{R}^2}(\tilde{C}_k^i) = 0$ for all $i < i_0$. Therefore, in both cases, if for some $i_1 < i_0$ and some subsequence in k it happens that

$$\lim_{k\to\infty} \operatorname{dist}_{\mathbb{R}^2}(\tilde{C}_k^{i_1}, \tilde{q}_k^{i_0}) = 0$$

or

$$\lim_{k\to\infty} \operatorname{dist}_{\mathbb{R}^2}(\tilde{C}_k^{i_1}, \tilde{q}_k^{i_0+1}) = 0,$$

then on the same subsequence

$$\lim_{k\to\infty} \operatorname{diam}_{\mathbb{R}^2}(\tilde{C}_k^i) = 0 \quad \text{for all } i \leq i_1. \tag{4.14}$$

Hence we are left to two possibilities:

Case (a): $\tilde{q}_\infty^i \neq (0,0), (1,0)$ (recall that $(0,0) = \tilde{q}_k^{i_0}$ and $(1,0) = \tilde{q}_k^{i_0+1}$ for every k). Then by the previous discussion, on the considered subsequence in k, we have

$$\begin{aligned} &\liminf_{k\to\infty} \operatorname{dist}_{\mathbb{R}^2}(\tilde{C}_k^i, \tilde{q}_k^{i_0}) > 0, \\ &\liminf_{k\to\infty} \operatorname{dist}_{\mathbb{R}^2}(\tilde{C}_k^i, \tilde{q}_k^{i_0+1}) > 0 \quad \text{for all } i < i_0. \end{aligned} \tag{4.15}$$

Case (b): for some $i_1 < i_0$ we have $\tilde{q}_\infty^{i_1} = (0,0)$ or $\tilde{q}_\infty^{i_1} = (1,0)$. Then, by the previous discussion, (4.14) holds.

Let us first consider case (a). The situation is analogous to Step 1 since for all $i < i_0$ the sets $\tilde{C}_k^i$ are at uniform strictly positive distance from $\tilde{q}_k^{i_0}$ and $\tilde{q}_k^{i_0+1}$. From the iterative construction of $\tilde{C}_k^i$ it is clear that, for every k, the set

$$\left(\mathbb{R}^2 \setminus \bigcup_{i<i_0} \bar{\tilde{C}}_k^i\right) \ni \{\tilde{q}_k^{i_0}, \tilde{q}_k^{i_0+1}\}$$

is connected and that it is possible to construct two smooth open subsets C^{i_0} and C^{i_0+1} such that

- $\bar{C}^{i_0}$ and $\bar{C}^{i_0+1}$ are diffeomorphic to closed balls,
- the intersection of $\bar{C}^{i_0}$ and $\bar{C}^{i_0+1}$ consists of just one point, say q_0:
$$\bar{C}^{i_0} \cap \bar{C}^{i_0+1} = \{q_0\} = \partial C^{i_0} \cap \partial C^{i_0+1}, \tag{4.16}$$
- $\tilde{q}_k^{i_0} = (0,0) \in C^{i_0}$ and $\tilde{q}_k^{i_0+1} = (1,0) \in C^{i_0+1}$,
- one has
$$\left(\partial C^{i_0} \cup \partial C^{i_0+1}\right) \cap \{a_1, \dots, a_N\} = \emptyset$$
and
$$\left(\partial C^{i_0} \cup \partial C^{i_0+1}\right) \cap \{\tilde{b}_k^1, \dots, \tilde{b}_k^{N_k}\} = \emptyset,$$
where $\{\tilde{b}_k^1, \dots, \tilde{b}_k^{N_k}\}$ are the branch points of $\tilde{\vec{\Phi}}_k = \vec{\Phi}_k \circ f_k$,
- $\liminf_{k\to\infty} \operatorname{dist}_{\mathbb{R}^2}(\tilde{q}_k^j, C^{i_0} \cup C^{i_0+1}) > 0$ for every $j > i_0 + 1$ (this can be done thanks to (4.12)),
- $\liminf_{k\to\infty} \operatorname{dist}_{\mathbb{R}^2}(\tilde{C}_k^i, C^{i_0} \cup C^{i_0+1}) > 0$ for all $i < i_0$ (this can be done thanks to (4.15)).

Now exactly as in Step 1, we have that

$$\mathcal{H}^1\left(\vec{\Phi}_k(\partial C^{i_0}) \cup \vec{\Phi}_k(\partial C^{i_0+1})\right) \to 0$$

and at least one between $\vec{\Phi}_k(C^{i_0})$ and $\vec{\Phi}_k(C^{i_0+1})$ has diameter at least $\frac{1}{2nC}$, say $\vec{\Phi}_k(C^{i_0})$:

$$\operatorname{diam}_M(\vec{\Phi}_k(C^{i_0})) \geq \frac{1}{2nC}.$$

As in Step 1 we can pass to the limit in inequality (4.9) and conclude that

$$4\pi - \frac{1}{n-1} \leq \int_{C^{i_0}} |\vec{H}_{\vec{I}} \circ \vec{\Phi}_k|^2 \, d\mathrm{vol}_{g_k} \quad \text{for large } k. \tag{4.17}$$

In other words, for large k we isolated another region $C^{i_0} \subset \mathbb{R}^2$ containing $q_k^{i_0}$, disjoint from the previous $\{\tilde{C}_k^i\}_{i<i_0}$, with Willmore energy at least $4\pi - \frac{1}{n-1}$. Observe that by construction (1a)–(1f) and (2) of the inductive step are satisfied.

Proof in case (b). Let us call

$$
\begin{aligned}
I_0 &:= \{i < i_0 : \tilde{q}^i_\infty = (0,0) = \tilde{q}^{i_0}_k\},\\
I_1 &:= \{i < i_0 : \tilde{q}^i_\infty = (1,0) = \tilde{q}^{i_0+1}_k\},\\
I_2 &:= \{1,\dots,i_0-1\} \setminus (I_0 \cup I_1);
\end{aligned}
$$

by assumption at least one between I_0 and I_1 is non-empty. Observe that, up to subsequences in k, we have

$$\forall i \in I_0 \quad \mathrm{dist}_{\mathbb{R}^2}(\tilde{C}^i_k, \tilde{q}^{i_0}_k) \to 0, \quad \mathrm{diam}_{\mathbb{R}^2}(\tilde{C}^i_k) \to 0, \tag{4.18}$$

$$\forall i \in I_1 \quad \mathrm{dist}_{\mathbb{R}^2}(\tilde{C}^i_k, \tilde{q}^{i_0+1}_k) \to 0, \quad \mathrm{diam}_{\mathbb{R}^2}(\tilde{C}^i_k) \to 0, \tag{4.19}$$

$$\forall i \in I_2 \quad \liminf_{k\to\infty} \mathrm{dist}_{\mathbb{R}^2}(\tilde{C}^i_k, \tilde{q}^{i_0}_k) > 0, \quad \liminf_{k} \mathrm{dist}_{\mathbb{R}^2}(\tilde{C}^i_k, \tilde{q}^{i_0+1}_k) > 0. \tag{4.20}$$

At first we do not consider I_0 and I_1 and construct C^{i_0} and C^{i_0+1} as in case (a) satisfying the same itemization with the only difference that in the last item we ask (it can be done thanks to (4.20))

$$\liminf_{k\to\infty} \mathrm{dist}_{\mathbb{R}^2}(\tilde{C}^i_k, C^{i_0} \cup C^{i_0+1}) > 0 \quad \text{for all } i \in I_2.$$

Now, for every k, call

$$C^{i_0}_k := C^{i_0} \setminus \left(\bigcup_{i\in I_0} \bar{\tilde{C}}^i_k \right) \quad \text{and} \quad C^{i_0+1}_k := C^{i_0+1} \setminus \left(\bigcup_{i\in I_1} \bar{\tilde{C}}^i_k \right). \tag{4.21}$$

Observe that, by (1a), $\tilde{q}^{i_0}_k = (0,0) \in C^{i_0}_k$ and $\tilde{q}^{i_0+1}_k = (1,0) \in C^{i_0+1}_k$ and, by the construction of C^{i_0}, C^{i_0+1} and since by (1b) we have

$$\mathcal{H}^1(\tilde{\vec{\Phi}}_k(\partial \tilde{C}^i_k)) = \mathcal{H}^1(\vec{\Phi}_k(\partial C^i_k)) \to 0 \quad \text{for all } i < i_0,$$

we still have that

$$\mathcal{H}^1(\tilde{\vec{\Phi}}_k(\partial C^{i_0}_k)) \to 0 \quad \text{and} \quad \mathcal{H}^1(\tilde{\vec{\Phi}}_k(\partial C^{i_0+1}_k)) \to 0. \tag{4.22}$$

Moreover, exactly as before, at least one between $\tilde{\vec{\Phi}}_k(C^{i_0}_k)$ and $\tilde{\vec{\Phi}}_k(C^{i_0+1}_k)$ has diameter at least $\frac{1}{2nC}$, say $\tilde{\vec{\Phi}}_k(C^{i_0}_k)$:

$$\mathrm{diam}_M(\vec{\Phi}_k(C^{i_0}_k)) \geq \frac{1}{2nC}. \tag{4.23}$$

Now observe that by construction of C^{i_0} and by assumption (1c), for every k we have

$$\partial C_k^{i_0} \cap \{\tilde{b}_k^1, \dots, \tilde{b}_k^{N_k}\} = \emptyset,$$

where $\{\tilde{b}_k^1, \dots, \tilde{b}_k^{N_k}\}$ are the branch points of $\tilde{\vec{\Phi}}_k$. Then, manipulating formula (4.8) as in Step 1 for the branched case, called $\{\tilde{b}_k^1, \dots, \tilde{b}_k^{L_k}\}$ the branch points contained in $C_k^{i_0}$ we obtain that

$$\begin{aligned} 4\pi \le & \int_{C_k^{i_0}} |\vec{H}_{\vec{I}} \circ \tilde{\vec{\Phi}}_k|^2 \, d\mathrm{vol}_{g_k} \\ & + 2 \frac{\mathcal{H}^1(\tilde{\vec{\Phi}}_k(\partial C_k^{i_0}))}{d(\vec{\Phi}_k(\partial C_k^{i_0}) \cup (\bigcup_{j=1}^{L_k} \tilde{\vec{\Phi}}_k(\tilde{b}_k^j)), \tilde{\vec{\Phi}}_k(C_k^{i_0}))}. \end{aligned} \tag{4.24}$$

As before, since $\mathrm{diam}_M(\tilde{\vec{\Phi}}_k(C_k^{i_0})) \ge \frac{1}{2nC}$ and $\mathcal{H}^1(\tilde{\vec{\Phi}}_k(\partial C_k^{i_0})) \to 0$, it follows that

$$\liminf_{k\to\infty} d\left(\vec{\Phi}_k(\partial C_k^{i_0}) \bigcup \left(\bigcup_{j=1}^{L_k} \tilde{\vec{\Phi}}_k(\tilde{b}_k^j) \right), \tilde{\vec{\Phi}}_k(C_k^{i_0}) \right) > 0,$$

and passing to the limit for $k \to \infty$ in (4.24), we get

$$4\pi - \frac{1}{n-1} \le \int_{C_k^{i_0}} |\vec{H}_{\vec{I}} \circ \tilde{\vec{\Phi}}_k|^2 \, d\mathrm{vol}_{g_k} \quad \text{for large } k. \tag{4.25}$$

In other words, for large k we isolated another region $C_k^{i_0} \subset \mathbb{R}^2$ containing $q_k^{i_0}$, disjoint from the previous $\{\tilde{C}_k^i\}_{i<i_0}$, with Willmore energy at least $4\pi - \frac{1}{n-1}$. Observe that by construction (1a)–(1f) and (2) are satisfied, so we completed the proof of the inductive step.

Let us summarize and conclude the proof: we showed that if the base of the induction is satisfied for a certain $2 \le i_0 \le n-1$, then we proved that the inductive step is true and so, from the procedure described above, it is clear that the base of induction is satisfied also for $i_0 + 1$. The iteration procedure stops for $i_0 = n-1$ and at that point, for large k, we constructed $n-1$ disjoint subsets $C_k^1, \dots, C_k^{n-1}$ each one carrying a Willmore energy of at least $4\pi - \frac{1}{n-1}$; then

$$\int_{\mathbb{S}^2} |d\vec{n}_{\vec{I}\circ\vec{\Phi}_k}|^2_{g_k} \, d\mathrm{vol}_{g_k} \ge \int_{\bigcup_{i=1}^{n-1} C_k^i} |d\vec{n}_{\vec{I}\circ\vec{\Phi}_k}|^2_{g_k} \, d\mathrm{vol}_{g_k} \ge 4\pi(n-1)-1. \tag{4.26}$$

Since n is arbitrary large, the lower bound (4.26) clearly contradicts the upper bound (4.1). $\square$

5 A diameter tracking procedure

The purpose of the present section is to prove the following lemma.

Lemma 5.1 (Diameter tracking procedure). *Let $\{\vec{\Phi}_k\}_{k\in\mathbb{N}} \subset \mathcal{F}_{\mathbb{S}^2}$ be a sequence of conformal weak, possibly branched, immersions into M^m. Assume that*

$$\limsup_{k\to+\infty} \int_{\mathbb{S}^2} \left[1 + |D\vec{n}_{\vec{\Phi}_k}|_h^2\right] d\,\mathrm{vol}_{g_k} < +\infty, \tag{5.1}$$

where $d\,\mathrm{vol}_{g_k}$ is the volume form associated to the induced metric $g_k := \vec{\Phi}_k^* h$ by $\vec{\Phi}_k$ on $\mathbb{S}^2$ and $\vec{H}_{\vec{\Phi}_k}$ is the mean curvature vector associated to the immersion $\vec{\Phi}_k$. Further, let $a \in \mathbb{S}^2$, $\delta_k \to 0$ and $\varepsilon_k \to 0$ and let a finite family of sequences of points $(a_k^i)_{i=1,\dots,N}$ together with a finite family of sequences of positive radii $(r_k^i)_{i=1,\dots,N}$ satisfy the following conditions:

$$\forall i \in \{1,\dots,N\} \qquad \lim_{k\to+\infty} \frac{|a_k^i - a|}{\varepsilon_k\,\delta_k} + \frac{r_k^j}{\varepsilon_k^2\,\delta_k} = 0, \tag{5.2}$$

$$\forall i \neq j \qquad \lim_{k\to+\infty} \frac{|a_k^i - a_k^j|}{\varepsilon_k^{-1}(r_k^i - r_k^j)} = 0, \tag{5.3}$$

$$\lim_{k\to+\infty} \int_{B_{\delta_k}(a)\setminus B_{\varepsilon_k\,\delta_k}(a)} \left[1 + |\mathbb{I}_{\vec{\Phi}_k}|_h^2\right] d\,\mathrm{vol}_{g_k} = 0, \tag{5.4}$$

$$\forall i \in \{1,\dots,N\} \qquad \lim_{k\to+\infty} \int_{B_{\varepsilon_k^{-1} r_k^i}(a_k^i)\setminus B_{r_k^i}(a_k^i)} \left[1 + |\mathbb{I}_{\vec{\Phi}_k}|_h^2\right] d\,\mathrm{vol}_{g_k} = 0. \tag{5.5}$$

Assume that $\vec{\Phi}_k$ has no branched points in each annulus $B_{\alpha^{-1}\, r_k^i}(a_k^i) \setminus B_{r_k^i}(a_k^i)$ as well as in $B_{\delta_k}(a) \setminus B_{\alpha\delta_k}(a)$ for any $0 < \alpha < 1$ and for k large enough. Suppose

$$\liminf_{k\to+\infty} \mathrm{diam}\left(\vec{\Phi}_k\left(B_{\varepsilon_k\,\delta_k}(a) \setminus \bigcup_{i=1}^{N} B_{\varepsilon_k^{-1} r_k^i}(a_k^i)\right)\right) > 0. \tag{5.6}$$

Then, modulo extraction of a subsequence, there exists a sequence of positive Möbius transformations $f_k \in \mathcal{M}^+(\mathbb{S}^2)$, a $Q \in \mathbb{N}$ and Q points $b_1,\dots,b_Q$ and $\vec{\xi}_\infty \in \mathcal{F}_{\mathbb{S}^2}$ such that

$$\vec{\xi}_k := \vec{\Phi}_k \circ f_k \rightharpoonup \vec{\xi}_\infty \quad \textit{weakly in } W^{2,2}_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{b_1,\dots,b_Q\}); \tag{5.7}$$

moreover for any compact $K \subset \mathbb{S}^2 \setminus \{b_1,\dots,b_Q\}$

$$\limsup_{k\to+\infty} \|\log|d(\vec{\Phi}_k \circ f_k)|\|_{L^\infty(K)} < +\infty. \tag{5.8}$$

Finally there exists $s_k \to 0$ *such that*

$$\vec{\Phi}_k \circ f_k \to \vec{\xi}_\infty \quad \textit{uniformly in } C^0 \textit{ on } \mathbb{S}^2 \setminus \bigcup_{j=1}^{Q} B_{s_k}(b_j), \tag{5.9}$$

and for any $i \in \{1, \dots, N\}$ *there exists* $j_i \in \{1, \dots, Q\}$ *such that*

$$f_k^{-1}(B_{r_k^i}(a_k^i)) \subset B_{s_k}(b_{j_i}); \tag{5.10}$$

moreover there exists also $j_0 \in \{1, \dots, Q\}$ *such that*

$$f_k^{-1}(\mathbb{S}^2 \setminus B_{\delta_k}(a)) \subset B_{s_k}(b_{j_0}). \tag{5.11}$$

Lemma 5.1 is a consequence of the combination of a so-called "Cutting and Filling Lemma" together with the "Good Gauge Extraction Lemma" 4.1. We first present the "Cutting and Filling Lemma" and its proof before to end this section with the proof of Lemma 5.1.

Lemma 5.2 (Cutting and Filling Lemma). *Let* $\{\vec{\Phi}_k\}_{k\in\mathbb{N}} \subset \mathcal{F}_{\mathbb{S}}^2$ *be a sequence of conformal weak, possibly branched, immersions* $\vec{\Phi}_k$ *of into* M^m. *Assume that*

$$\limsup_{k\to+\infty} \int_{\mathbb{S}^2} \left[1 + |D\vec{n}_{\vec{\Phi}_k}|_h^2\right] d\,\mathrm{vol}_{g_k} < +\infty, \tag{5.12}$$

where $d\,\mathrm{vol}_{g_k}$ *is the volume form associated to the induced metric* $g_k := \vec{\Phi}_k^* h$ *by* $\vec{\Phi}_k$ *on* $\mathbb{S}^2$ *and* $\vec{H}_{\vec{\Phi}_k}$ *is the mean curvature vector associated to the immersion* $\vec{\Phi}_k$. *Let* $a \in \mathbb{S}^2$ *and* $s_k, t_k \to 0$ *be such that*

$$\frac{t_k}{s_k} \to 0, \tag{5.13}$$

and

$$\lim_{k\to+\infty} \int_{B_{s_k}(a)\setminus B_{t_k}(a)} \left[1 + |\mathbb{I}_{\vec{\Phi}_k}|_h^2\right] d\,\mathrm{vol}_{g_k} = 0 \tag{5.14}$$

and assume that $\vec{\Phi}_k$ *has no branch points in* $B_{s_k}(a) \setminus B_{\alpha\, s_k}(a)$ *for any* $0 < \alpha < 1$ *and* k *large enough. Then there exists a conformal immersion* $\vec{\xi}_k$ *from* $\mathbb{S}^2$ *into* M^m *and a sequence of quasi-conformal bilipschitz homeomorphisms* Ψ_k *of* $\mathbb{S}^2$, *converging in* C^0*-norm over* $\mathbb{S}^2$ *to the identity map, such that*

$$\vec{\xi}_k \circ \Psi_k = \vec{\Phi}_k \quad \textit{in } \mathbb{S}^2 \setminus B_{s_k}(a)$$

and

$$\lim_{k\to+\infty} \mathrm{diam}(\vec{\xi}_k \circ \Psi_k(B_{s_k}(a)) = 0, \qquad \lim_{k\to+\infty} \mathrm{Area}(\vec{\xi}_k \circ \Psi_k(B_{s_k}(a)) = 0;$$

moreover

$$\lim_{k\to+\infty}\int_{B_{s_k}(a)}\left[1+|\mathbb{I}^0_{\vec{\xi}_k\circ\Psi_k}|^2_h\right]d\mathrm{vol}_{g_{\vec{\xi}_k\circ\Psi_k}}=0, \tag{5.15}$$

where $\mathbb{I}^0_{\vec{\xi}_k\circ\Psi_k}$ *is the trace-free second fundamental form.*

Proof. For simplicity of the presentation we prove the lemma for a sequence of conformal immersions $\vec{\Phi}_k$ into the unit ball of $\mathbb{R}^m$ since the proof for immersions into M^m is fundamentally of identical nature. We apply successively the stereographic projection π from $\mathbb{S}^2$ into $\mathbb{C}$ that sends a to the origin 0 of the complex plane and the dilation of radius $\simeq 1/s_k$ such that the composition of these two maps is sending $B_{s_k}(a)\setminus B_{t_k}(a)$ into $B_1(0)\setminus B_{\rho_k}(0)$ with $\rho_k\to 0$. We keep denoting $\vec{\Phi}_k$ the conformal map that we obtained after composing the original $\vec{\Phi}_k$ with these two maps. Since for any $1>\alpha>0$ and k large enough $\vec{\Phi}_k$ realizes a Lipschitz conformal immersion of the annulus $B_1(0)\setminus B_\alpha(0)$ into the unit ball of $\mathbb{R}^m$, there exists $\lambda_k\in L^\infty(B_1(0)\setminus B_\alpha(0))$ such that

$$e^{\lambda_k}=|\partial_{x_1}\vec{\Phi}_k|=|\partial_{x_2}\vec{\Phi}_k|\quad\text{in } B_1(0)\setminus B_\alpha(0).$$

From (5.14) we deduce that

$$\begin{aligned}&\lim_{k\to+\infty}\int_{B_1(0)\setminus B_{\rho_k}(0)}|d\vec{n}_{\vec{\Phi}_k}|^2_{g_k}\,d\mathrm{vol}_{g_k}\\&\quad=\lim_{k\to+\infty}\int_{B_1(0)\setminus B_{\rho_k}(0)}|\nabla\vec{n}_{\vec{\Phi}_k}|^2\,dx=0.\end{aligned}\tag{5.16}$$

Using the argument in [4] – first by [4, Lemma VI.1] we extend $\vec{n}_{\Phi_k}$ in B_{ρ_k} with energy control and then we use Hélein's construction of energy controlled moving frame, see [8, Theorem V.2.1] – we deduce the existence of an orthonormal frame $(\vec{e}_{1,k},\vec{e}_{2,k})$ on $B_1(0)\setminus B_{\rho_k}(0)$ such that

$$\star(\vec{e}_{1,k}\wedge\vec{e}_{2,k})=\vec{n}_{\vec{\Phi}_k}$$

and

$$\int_{B_1(0)\setminus B_{\rho_k}(0)}\sum_{i=1}^2|\nabla\vec{e}_{i,k}|^2\,dx\le C\int_{B_1(0)\setminus B_{\rho_k}(0)}|\nabla\vec{n}_{\vec{\Phi}_k}|^2\,dx. \tag{5.17}$$

A classical computation (see [8] and [20]) gives

$$\Delta\lambda_k=\nabla^\perp\vec{e}_{1,k}\cdot\nabla\vec{e}_{2,k}. \tag{5.18}$$

Let μ_k satisfy

$$\begin{cases} \Delta\mu_k = \nabla^\perp \vec{e}_{1,k} \cdot \nabla \vec{e}_{2,k} & \text{in } B_1(0) \setminus B_\alpha(0), \\ \quad \mu_k = 0 & \text{on } \partial(B_1(0) \setminus B_\alpha(0)). \end{cases} \tag{5.19}$$

Wente's inequality (see [6,8,25]) gives the existence of a constant $C > 0$ independent of α such that

$$\begin{aligned} \|\mu_k\|_{L^\infty(B_1(0)\setminus B_\alpha(0)} &+ \|\nabla\mu_k\|_{L^2(B_1(0)\setminus B_\alpha(0)} \\ &\leq C\ \|\nabla e_{1,k}\|_{L^2}\ \|\nabla e_{2,k}\|_{L^2} \\ &\leq C' \int_{B_1(0)\setminus B_{\rho_k}(0)} |\nabla \vec{n}_{\vec{\Phi}_k}|^2\, dx. \end{aligned} \tag{5.20}$$

Since from Lemma 2.2 we know that $\nabla\lambda_k$ is uniformly bounded in $L^{2,\infty}$, using (5.20), we have that the harmonic function $\nu_k := \lambda_k - \mu_k$ satisfies

$$\limsup_{k\to+\infty} \|\nabla\nu_k\|_{L^{2,\infty}(B_1(0)\setminus B_\alpha(0))} < +\infty. \tag{5.21}$$

Standard elliptic estimates on harmonic function (see for instance [7]) imply that for any $2\alpha < \delta < 1$

$$\limsup_{k\to+\infty} \|\nu_k(x) - \nu_k(y)\|_{L^\infty((B_\delta(0)\setminus B_{2\alpha}(0))^2)} < +\infty.$$

Combining this fact with (5.20) we obtain that there exists a constant $\overline{\lambda}_k$ satisfying

$$\lim_{k\to+\infty} \overline{\lambda}_k = -\infty, \tag{5.22}$$

and such that for any choice $2\alpha < \delta < 1$

$$\limsup_{k\to+\infty} \|\lambda_k - \overline{\lambda}_k\|_{L^\infty(B_\delta(0)\setminus B_{2\alpha}(0))}. \tag{5.23}$$

We introduce the new map given by

$$\hat{\vec{\Phi}}_k := e^{-\overline{\lambda}_k}\left[\vec{\Phi}_k - \vec{\Phi}_k\left(0, \frac{1}{2}\right)\right].$$

Because of (5.23), it is clear that

$$\hat{\vec{\Phi}}_k(B_\delta(0) \setminus B_{2\alpha}(0)) \subset B_{R_{\delta,\alpha}}(0) \quad \text{for some } R_{\delta,\alpha} > 0$$

and that $\hat{\vec{\Phi}}_k$ converges weakly in $(W^{1,\infty})^*$ to some non-trivial limiting conformal immersion $\hat{\vec{\Phi}}_\infty$ of $B_\delta(0) \setminus B_{2\alpha}(0)$:

$$\hat{\vec{\Phi}}_k \rightharpoonup \hat{\vec{\Phi}}_\infty \quad \text{weakly in } (W^{1,\infty})^*(B_\delta(0) \setminus B_{2\alpha}(0)). \tag{5.24}$$

Denote

$$\hat{\lambda}_k := \log|\partial_{x_1}\hat{\vec{\Phi}}_k| = \log|\partial_{x_2}\hat{\vec{\Phi}}_k|.$$

Since $\hat{\lambda}_k$ is uniformly bounded in $L^\infty(B_\delta(0) \setminus B_{2\alpha}(0))$, since $\hat{\vec{\Phi}}_k$ satisfies

$$\Delta\hat{\vec{\Phi}}_k = \frac{e^{2\hat{\lambda}_k}}{2}\vec{H}_{\hat{\vec{\Phi}}_k}$$

and since the L^2-norm of $\vec{H}_{\hat{\vec{\Phi}}_k}$ is uniformly bounded (due to (5.16)), we deduce that

$$\hat{\vec{\Phi}}_k \rightharpoonup \hat{\vec{\Phi}}_\infty \quad \text{weakly in } W^{2,2}(B_\delta(0) \setminus B_{2\alpha}(0)). \tag{5.25}$$

From (5.16) we have that for any $2\alpha < \delta < 1$

$$\int_{B_\delta(0)\setminus B_{2\alpha}(0)} |\nabla\vec{n}_{\hat{\vec{\Phi}}_\infty}|^2\, dx = 0.$$

This implies that $\vec{n}_{\hat{\vec{\Phi}}_\infty}$ is constant on $D^2 \setminus \{0\}$ equal to a unit simple $m-2$ vector $\vec{n}_0$, i.e.

$$\vec{n}_0 = \vec{v}_1 \wedge \cdots \wedge \vec{v}_{m-2}$$

for some constant vectors $\vec{v}_1, \ldots, \vec{v}_{m-2}$ of $\mathbb{R}^m$. Thus $\hat{\vec{\Phi}}_\infty$ is conformal from $D^2 \setminus \{0\}$ into a two-dimensional plane P_0^2 that we identify to $z_i = 0$ for $i \geq 3$ and identifies to an holomorphic map f_∞. Denote $\hat{\lambda}_\infty$ the limit of $\hat{\lambda}_k$

$$\hat{\lambda}_\infty = \log|\partial_{x_1}\hat{\vec{\Phi}}_\infty| = \log|\partial_{x_2}\hat{\vec{\Phi}}_\infty| = \log|f'(x_1 + ix_2)|,$$

which is harmonic in $D^2 \setminus \{0\}$ – this can be deduced from (5.16)–(5.18). Since again from Lemma 2.2 $\nabla\lambda_k$ is uniformly bounded in $L^{2,\infty}$, we have

$$\|\nabla\hat{\lambda}_\infty\|_{L^{2,\infty}} < +\infty. \tag{5.26}$$

Thus there exists a $c_0 \in \mathbb{R}$ such that

$$\Delta\hat{\lambda}_\infty = c_0\,\delta_0, \tag{5.27}$$

and $c_0 = 2\pi\,(\theta_0 - 1) \in 2\pi\mathbb{Z}$, where θ_0 is the order of the zero of the pole of f_∞ at 0. Consider the 2-sphere S_k^2 of radius $e^{-\overline{\lambda}_k/2}$ tending to infinity and tangent to the 2-plane P_0^2 at 0 and given by $z_i = 0$ for $i \geq 3$ at 0 and contained in the half 3-space E_+^3 given by $z_i = 0$ for $i \geq 4$ and $z_3 \geq 0$.

Let π_k be the stereographic projection from S_k^2 into the 2-plane P_0^2 such that

$$\pi_k(0) = (0, \ldots, 0)$$

and

$$\pi_k(0, 0, 2\, e^{-\overline{\lambda}_k/2}, 0, \ldots, 0) = \infty.$$

It is clear that for any $R > 0$

$$\|\pi_k^{-1}(z) - z\|_{C^l(B_R^2(0))} \leq C_{R,l}\, e^{\overline{\lambda}_k/2} \quad \text{for all } l \in \mathbb{N},$$

where $B_R^2(0) := B_R^m(0) \cap P_0^2$. Hence, for any choice $2\alpha < \delta < 1$, we have

$$\lim_{k\to+\infty} \|\pi_k^{-1} \circ \hat{\vec{\Phi}}_\infty - \hat{\vec{\Phi}}_\infty\|_{C^l(B_\delta(0)\setminus B_{2\alpha}(0))} \to 0. \tag{5.28}$$

The advantage of considering $\pi_k^{-1} \circ \hat{\vec{\Phi}}_\infty$ instead of $\hat{\vec{\Phi}}_\infty$ is that $\pi_k^{-1} \circ \hat{\vec{\Phi}}_\infty(D^2)$ is compact even if $\theta_0 < 0$ and since π_k^{-1} is conformal, $\pi_k^{-1} \circ \hat{\vec{\Phi}}_\infty$ is also conformal. Using (5.25) we have for any choice of $2\alpha < \delta < 1$

$$\hat{\vec{\Phi}}_k - \pi_k^{-1} \circ \hat{\vec{\Phi}}_\infty \rightharpoonup 0 \quad \text{weakly in } (W^{1,\infty})^* \cap W^{2,2}(B_\delta(0) \setminus B_{2\alpha}(0)). \tag{5.29}$$

Using Fubini's theorem together with the mean value formula, we can find a "good radius" $r_k \in (1/2, 1)$ such that

$$\limsup_{k\to+\infty} \int_{\partial B_{r_k}} |\nabla^2 \hat{\vec{\Phi}}_k|^2\, dl < +\infty, \tag{5.30}$$

where dl is the length form on ∂B_{r_k}. Because of (5.29), we have that

$$\hat{\vec{\Phi}}_k - \pi_k^{-1} \circ \hat{\vec{\Phi}}_\infty \rightharpoonup 0 \quad \text{weakly in } H^{3/2}(\partial B_{r_k}) \tag{5.31}$$

and

$$\partial_r \hat{\vec{\Phi}}_k - \partial_r(\pi_k^{-1} \circ \hat{\vec{\Phi}}_\infty) \rightharpoonup 0 \quad \text{weakly in } H^{1/2}(\partial B_{r_k}). \tag{5.32}$$

Combining (5.30)–(5.32) we deduce

$$\hat{\vec{\Phi}}_k - \pi_k^{-1} \circ \hat{\vec{\Phi}}_\infty \rightharpoonup 0 \quad \text{weakly in } W^{2,2}(\partial B_{r_k}) \tag{5.33}$$

and

$$\partial_r \hat{\vec{\Phi}}_k - \partial_r(\pi_k^{-1} \circ \hat{\vec{\Phi}}_\infty) \rightharpoonup 0 \quad \text{weakly in } W^{1,2}(\partial B_{r_k}). \tag{5.34}$$

Consider the map solving

$$\begin{cases} \Delta^2 \tilde{\vec{\Phi}}_k = 0 & \text{in } B_{r_k} \setminus B_{r_k/2}, \\ \tilde{\vec{\Phi}}_k = \hat{\vec{\Phi}}_k & \text{in } \partial B_{r_k}, \\ \tilde{\vec{\Phi}}_k = \pi_k^{-1} \circ \hat{\vec{\Phi}}_\infty & \text{in } \partial B_{r_k/2}, \\ \partial_r \tilde{\vec{\Phi}}_k = \partial_r \hat{\vec{\Phi}}_k & \text{in } \partial B_{r_k}, \\ \partial_r \tilde{\vec{\Phi}}_k = \partial_r (\pi_k^{-1} \circ \hat{\vec{\Phi}}_\infty) & \text{in } \partial B_{r_k/2}. \end{cases} \tag{5.35}$$

Since $\hat{\vec{\Phi}}_\infty$ is holomorphic, and hence biharmonic, combining (5.28), (5.33)–(5.35) together with classical elliptic estimates we obtain

$$\tilde{\vec{\Phi}}_k - \hat{\vec{\Phi}}_\infty \rightharpoonup 0 \quad \text{weakly in } W^{5/2,2}(B_{r_k} \setminus B_{r_k/2}), \tag{5.36}$$

which implies, by Sobolev embeddings,

$$\lim_{k \to 0} \|\tilde{\vec{\Phi}}_k - \hat{\vec{\Phi}}_\infty\|_{C^{1,\alpha}(B_{r_k} \setminus B_{r_k/2})} = 0 \quad \text{for any } \alpha < \frac{1}{2}. \tag{5.37}$$

We extend $\tilde{\vec{\Phi}}_k$ by $\pi_k^{-1} \circ \hat{\vec{\Phi}}_\infty$ in $B_{r_k/2}$ and by $\hat{\Phi}_k$ in the complement of B_{r_k}. Finally we denote

$$\vec{\zeta}_k := e^{\overline{\lambda}_k} \, \tilde{\vec{\Phi}}_k \left(\frac{x}{s_k} \right) + \vec{\Phi}_k \left(0, \frac{1}{2} \right)$$

in such a way that:

$$\vec{\zeta}_k \equiv \vec{\Phi}_k \quad \text{in } \mathbb{C} \setminus B_{s_k}, \tag{5.38}$$

$$\lim_{k \to 0} \operatorname{diam}(\vec{\zeta}_k(B_{s_k}(0))) = 0, \tag{5.39}$$

$$\lim_{k \to +\infty} \int_{B_{s_k}(0)} \left[1 + |\mathbb{I}^0_{\vec{\zeta}_k}|^2 \right] d\mathrm{vol}_{g_{\vec{\zeta}_k}} = 0, \tag{5.40}$$

and finally introducing

$$\sigma_k := \frac{g^{11}_{\vec{\zeta}_k} - g^{22}_{\vec{\zeta}_k} - 2\,i\, g^{12}_{\vec{\zeta}_k}}{g^{11}_{\vec{\zeta}_k} + g^{22}_{\vec{\zeta}_k} + \sqrt{\det g_{\vec{\zeta}_k}}}, \tag{5.41}$$

where $g^{ij}_{\vec{\zeta}_k} := \partial_{x_i} \vec{\zeta}_k \cdot \partial_{x_j} \vec{\zeta}_k$, using (5.36) and (5.37), we have that

$$\operatorname{supp}(\sigma_k) \subset B_{s_k r_k} \setminus B_{s_k r_k/2}, \qquad \lim_{k \to +\infty} \|\sigma_k\|_{L^\infty(\mathbb{C})} + \|\nabla \sigma_k\|_{L^2(\mathbb{C})} = 0. \tag{5.42}$$

Let $\psi_k(z) := \phi_k(z) + z$, where ϕ_k is the fixed point in $\dot{H}^1(\mathbb{C})$ given by

$$\begin{cases} \phi_k(z) + \dfrac{1}{\pi z} * (\sigma_k\ \partial_z \phi_k) = -\dfrac{1}{\pi z} * \sigma_k, \\ \phi_k(0) = 0. \end{cases}$$

Notice that ϕ_k satisfies then

$$\partial_{\overline{z}} \phi_k = \sigma_k\ \partial_z \phi_k + \sigma_k. \tag{5.43}$$

and

$$\|\nabla \phi_k\|_{L^2(\mathbb{C})} \leq C\ \|\sigma_k\|_{L^2(\mathbb{C})} \to 0. \tag{5.44}$$

It is a classical fact from quasi-conformal mapping theory and from the classical analysis of Beltrami equation that ψ_k realizes an Hölder homeomorphism from $\mathbb{C} \cup \{\infty\}$ into $\mathbb{C} \cup \{\infty\}$ (see for instance [9, Section 4.2]), moreover there exists a $p > 2$ such that

$$\|\nabla \phi_k\|_{L^p(\mathbb{C})} \leq C\ \|\sigma_k\|_{L^p(\mathbb{C})} \to 0. \tag{5.45}$$

This implies in particular that

$$\|\psi_k(x) - x\|_{C^{0,\alpha}} \to 0 \tag{5.46}$$

for some $0 < \alpha < 1$ and thus

$$\psi_k\big(B_{s_k\, r_k}(0) \setminus B_{s_k\, r_k/2}(0)\big) \subset B_{2(s_k\, r_k)^\alpha}(0) \tag{5.47}$$

for k large enough. A classical computation gives

$$\sigma_{\vec{\xi}_k \circ \psi_k^{-1}} = \frac{g^{11}_{\vec{\xi}_k \circ \psi_k^{-1}} - g^{22}_{\vec{\xi}_k \circ \psi_k^{-1}} - 2\, i\ g^{12}_{\vec{\xi}_k \circ \psi_k^{-1}}}{g^{11}_{\vec{\xi}_k \circ \psi_k^{-1}} + g^{22}_{\vec{\xi}_k \circ \psi_k^{-1}} + \sqrt{\det g_{\vec{\xi}_k \circ \psi_k^{-1}}}} = 0, \tag{5.48}$$

where

$$g^{ij}_{\vec{\xi}_k \circ \psi_k^{-1}} := \partial_{x_i}(\vec{\xi}_k \circ \psi_k^{-1}) \cdot \partial_{x_j}(\vec{\xi}_k \circ \psi_k^{-1}).$$

Again classical computations (see [9, Section 4.2]) gives that ψ_k^{-1} is the *normal solution* to

$$\partial_{\overline{w}} \psi_k^{-1} = -\sigma_k \circ \psi_k^{-1}\, \partial_w\, \psi_k^{-1},$$

where $w = y_1 + i y_2 \in \mathbb{C}$ (i.e. ψ_k is the unique continuous function satisfying the above identity and such that $\psi_k(0) = 0$ and such that $\partial_w \psi_k - 1$ belongs to $L^p(\mathbb{C})$, for some $p > 2$; for more details see [9, Theorem 4.24]). Moreover there exists a φ_k such that $\psi_k^{-1}(y) = y + \varphi_k(y)$ and, because of (5.47), we have that

$$\mathrm{supp}(-\sigma_k \circ \psi_k^{-1}) \subset B_{2(s_k\, r_k)^\alpha}(0), \qquad \lim_{k\to+\infty} \|-\sigma_k \circ \psi_k^{-1}\|_{L^\infty(\mathbb{C})} = 0.$$

Thus, as before, we have the existence of $p > 2$ such that

$$\|\nabla\varphi_k\|_{L^2(\mathbb{C})} + \|\nabla\varphi_k\|_{L^p(\mathbb{C})} \to 0. \tag{5.49}$$

We now go back to the sphere $\mathbb{S}^2$ and we set

$$\begin{cases} \vec{\xi}_k := \vec{\zeta}_k \circ \psi_k^{-1} \circ \pi, \\ \Psi_k := \pi^{-1} \circ \psi_k \circ \pi, \end{cases} \tag{5.50}$$

where we recall that π is the stereographic projection from $\mathbb{S}^2$ into $\mathbb{C}$ that sends $a \in \mathbb{S}^2$ into 0. Because of (5.46), Ψ_k converges uniformly to the identity in any compact of $\mathbb{S}^2 \setminus \{-a\}$. Moreover, still because of (5.46), for any $\varepsilon > 0$ there exist k_0 and δ such that for all $k > k_0$ and $r < \delta$,

$$\|\Psi_k(x) + a\|_{L^\infty(B_r(-a))} < \varepsilon.$$

Thus Ψ_k converges uniformly to the identity on $\mathbb{S}^2$. Finally combining this fact with (5.38)–(5.40) we have proved the "Cutting and Filling Lemma" 5.2. □

Proof of Lemma 5.1. Let $\{\vec{\Phi}_k\}_{k\in\mathbb{N}}$ denote a sequence of conformal, possibly branched, weak immersions in $\mathcal{F}_{\mathbb{S}^2}$ satisfying the assumption of the lemma. Denote by π the stereographic projection that sends a to zero and which is almost an isometry from small geodesic balls centred at a into the corresponding Euclidean ball centred at 0. We have for instance that $\pi(B_{r_k^i}(a_k^i))$ (resp. $\pi(B_{\varepsilon_k^{-1} r_k^i}(a_k^i))$ is "almost" the ball of centre

$$x_k^i := \pi(a_k^i)$$

and radius r_k^i (resp. radius $\varepsilon_k^{-1} r_k^i$). In order to simplify the presentation we will identify

- $\pi(B_{r_k^i}(a_k^i))$ with $B^2_{r_k^i}(x_k^i)$,
- $\pi(B_{\varepsilon_k^{-1} r_k^i}(a_k^i))$ with $B^2_{\varepsilon_k^{-1} r_k^i}(x_k^i)$,
- $\pi(B_{\delta_k}(a))$ with $B^2_{\delta_k}(0)$,
- $\pi(B_{\varepsilon_k\,\delta_k}(a))$ with $B^2_{\varepsilon_k\,\delta_k}(0)$

in the list of assumptions going from (5.2) until (5.5).

Let

$$D_k(x) := \frac{x}{\varepsilon_k \delta_k}$$

be the dilation in $\mathbb{C}$ of factor $(\varepsilon_k\delta_k)^{-1}$. We consider the new conformal immersion $\tilde{\vec{\Phi}}_k$ given by

$$\tilde{\vec{\Phi}}_k := \vec{\Phi}_k \circ \pi^{-1} \circ D_k \circ \pi.$$

We can apply Lemma 5.2 $N+1$ times respectively in the ball

$$\pi^{-1}(\mathbb{C} \setminus B_{1/\sqrt{\varepsilon_k}}(0)) := B_{s_k}(-a)$$

and in the balls

$$\pi^{-1}(B_{r_k^i/(\varepsilon_k^2\delta_k)}(x_k^i/(\varepsilon_k\delta_k))) := B_{r_k^i/(\varepsilon_k^2\delta_k)}(c_k^i),$$

where

$$\rho_i^k := \frac{r_k^i}{\varepsilon_k^2\delta_k} \to 0 \quad \text{and} \quad c_k^i := \pi^{-1}(x_k^i/(\varepsilon_k\delta_k)) \to a.$$

We then generate a sequence $\{\Psi_k\}_{k\in\mathbb{N}}$ of bilipschitz quasi-conformal homeomorphisms of $\mathbb{S}^2$ converging uniformly to the identity (with a distortion μ_{Ψ_k} converging uniformly to zero) and a sequence $\vec{\xi}_k$ of conformal, possibly branched, weak immersions in $\mathcal{F}_{\mathbb{S}^2}$ satisfying

$$\vec{\xi}_k \circ \Psi_k = \tilde{\vec{\Phi}}_k \quad \text{in } \mathbb{S}^2 \setminus B_{s_k}(-a) \bigcup_{i=1}^N B_{\rho_k^i}(c_k^i). \tag{5.51}$$

Moreover

$$\lim_{k\to+\infty} \operatorname{diam}\big(\vec{\xi}_k \circ \Psi_k(B_{s_k}(-a))\big) + \sum_{i=1}^N \operatorname{diam}\big(\vec{\xi}_k \circ \Psi_k(B_{\rho_k^i}(c_k^i))\big) = 0, \tag{5.52}$$

whereas, from the assumption (5.6) one has

$$\liminf_{k\to+\infty} \operatorname{diam}\left(\vec{\xi}_k \circ \Psi_k\left(\mathbb{S}^2 \setminus \left(B_{s_k}(-a) \bigcup_{i=1}^N B_{\rho_k^i}(c_k^i)\right)\right)\right) > 0. \tag{5.53}$$

We apply the "Good Gauge Extraction Lemma" 4.1 to the immersion $\vec{\xi}_k$ and we obtain $u_k \in \mathcal{M}^+(\mathbb{S}^2)$, an element $\vec{\zeta}_\infty \in \mathcal{F}_{\mathbb{S}^2}$ and Q points $b_1, \dots, b_Q$ such that

$$\vec{\zeta}_k := \vec{\xi}_k \circ u_k \rightharpoonup \vec{\zeta}_\infty \quad \text{weakly in } W^{2,2}(\mathbb{S}^2 \setminus \{b_1, \dots, b_Q\}). \tag{5.54}$$

Because of (5.52), there exist $p_0, p_1, \dots, p_N \in M^m$ such that

$$\vec{\xi}_k \circ \Psi_k(B_{s_k}(-a)) \to p_0$$

and

$$\vec{\xi}_k \circ \Psi_k(B_{\rho_k^i}(c_k^i)) \to p_i \quad \text{for all } i \in \{1, \dots, N\}.$$

Since the Willmore energy of $\vec{\zeta}_\infty$ is finite, each point admits a controlled number of preimages. Denote by

$$\{x_1, \dots, x_L\} = \bigcup_{i=0}^{N} \vec{\zeta}_\infty^{-1}(\{p_i\}).$$

For any $\varepsilon > 0$ and for k large enough

$$\zeta_k\left(\mathbb{S}^2 \setminus \bigcup_{l=1}^{L} B_\varepsilon(x_l)\right) \cap \left(\vec{\xi}_k \circ \Psi_k(B_{s_k}(-a)) \bigcup_{i=1}^{N} \vec{\xi}_k \circ \Psi_k(B_{\rho_k^i}(c_k^i))\right) = \emptyset.$$

Thus for any ε and k large enough

$$\vec{\zeta}_k(x) = \tilde{\vec{\Phi}}_k \circ \Psi_k^{-1} \circ u_k(x) \quad \text{for all } x \in \mathbb{S}^2 \setminus \bigcup_{l=1}^{L} B_\varepsilon(x_l). \tag{5.55}$$

Consider the following sequence of quasi-conformal homeomorphisms of the sphere,

$$\Lambda_k := \Psi_k^{-1} \circ u_k.$$

The identity (5.55) implies that Λ_k is conformal on $\mathbb{S}^2 \setminus \bigcup_{l=1}^{L} B_\varepsilon(x_l)$ for k large enough. Let π be a fixed stereographic projection that sends none of the points x_l to infinity: for ε small enough $\pi(\bigcup_{l=1}^{L} B_\varepsilon(x_l))$ is sent into a fixed ball $B_R(0)$ of the complex plane. Let

$$\mu_{\Lambda_k \circ \pi^{-1}} := \frac{\partial_{\overline{z}}(\Lambda_k \circ \pi^{-1})}{\partial_z(\Lambda_k \circ \pi^{-1})}.$$

We have then from the previous consideration and the proof of Lemma 5.2

$$\operatorname{supp}(\mu_{\Lambda_k \circ \pi^{-1}}) \subset B_R(0) \quad \text{and} \quad \|\mu_{\Lambda_k \circ \pi^{-1}}\|_{L^\infty(\mathbb{C})} \to 0. \tag{5.56}$$

Let ψ_k be the normal solution (see again [9, Chapter 4]) of

$$\begin{cases} \partial_{\overline{z}} \psi_k = \mu_{\Lambda_k \circ \pi^{-1}} \partial_z \psi_k & \text{on } \mathbb{C}, \\ \psi_k(0) = 0, \end{cases}$$

such that there exists a $p > 2$ for which $\nabla(\psi_k(x) - x) \in L^p(\mathbb{C})$. It is a classical result, since (5.56) holds, that $\{\psi_k\}_{k \in \mathbb{N}}$ and $\{\psi_k^{-1}\}_{k \in \mathbb{N}}$ are compact in $C^{0,\alpha}$ for some $\alpha < 1$ (see [9, 4.2.3]), and converge to the identity. Moreover, a classical computation (see [9, Proposition 4.13]) gives that $\Lambda_k \circ \pi^{-1} \circ \psi_k^{-1} \circ \pi$ is conformal.

Consider $\zeta_k \circ \pi^{-1} \circ \psi_k^{-1} \circ \pi$. This sequence of maps converges in $C^{0,\alpha}$-norm to $\vec{\zeta}_\infty$ and in particular we have

$$\tilde{\vec{\tilde{\Phi}}}_k \circ \Lambda_k \circ \pi^{-1} \circ \psi_k^{-1} \circ \pi \to \vec{\zeta}_\infty \quad \text{in } C^{0,\alpha}\left(\mathbb{S}^2 \setminus \bigcup_{l=1}^{L} B_\varepsilon(x_l)\right). \tag{5.57}$$

We apply now Proposition 3.1 to $\tilde{\vec{\tilde{\Phi}}}_k \circ \Lambda_k \circ \pi^{-1} \circ \psi_k^{-1} \circ \pi$ which is conformal on $\mathbb{S}^2$. Assume that, modulo extraction of a subsequence,

$$\tilde{\vec{\tilde{\Phi}}}_k \circ \Lambda_k \circ \pi^{-1} \circ \psi_k^{-1} \circ \pi$$

would converge uniformly to a point on any compact $K \subset \mathbb{S}^2 \setminus \{a_1, \dots, a_P\}$ for some finite collection of points $\{a_1, \dots, a_P\}$; this would contradict (5.57). Thus we can take

$$f_k := \pi^{-1} \circ D_k \circ \pi \circ \Lambda_k \circ \pi^{-1} \circ \psi_k^{-1} \circ \pi,$$

and one easily checks that the required conditions (5.7)–(5.11) are fulfilled for this choice of f_k and Lemma 5.1 is proved. $\square$

6 Domain decomposition and the proof of Theorem 1.1

Before moving to the proof of Theorem 1.1 we will first establish the following lemma.

Lemma 6.1 (Domain Decomposition Lemma). *Let $\{\vec{\Phi}_k\}_{k\in\mathbb{N}} \subset \mathcal{F}_{\mathbb{S}}^2$ be a sequence of conformal weak, possibly branched, immersion into M^m. Assume that*

$$\limsup_{k\to+\infty} \int_{\mathbb{S}^2} \left[1 + |D\vec{n}_{\vec{\Phi}_k}|_h^2\right] d\,\mathrm{vol}_{g_k} < +\infty, \tag{6.1}$$

where $d\,\mathrm{vol}_{g_k}$ is the volume form associated to the induced metric $g_k := \vec{\Phi}_k^* h$ by $\vec{\Phi}_k$ on $\mathbb{S}^2$ and $\vec{H}_{\vec{\Phi}_k}$ is the mean curvature vector associated to the immersion $\vec{\Phi}_k$. Then, modulo extraction of a subsequence, there exists an $N \in \mathbb{N}$, there exist N sequences of Möbius transformations in $\mathcal{M}^+(\mathbb{S}^2)$, $(f_k^i)_{i=1,\dots,N}$, N elements of $\mathcal{F}_{\mathbb{S}^2}$, $(\vec{\xi}_\infty^i)_{i=1,\dots,N}$, and N natural integers $(N_i)_{i=1,\dots,N}$ such that

$$\vec{\Phi}_k \circ f_k^i \rightharpoonup \vec{\xi}_\infty^i \quad \text{weakly in } W^{2,2}(\mathbb{S}^2 \setminus \{b^{i,1}, \dots, b^{i,N^i}\}), \tag{6.2}$$

where, for each $i \in \{1, \dots, N\}$, $(b^{i,j})_{j=1,\dots,N^i}$ is a finite family of points in $\mathbb{S}^2$. There exists a sequence $s_k \to 0$ such that for any $i = 1, \dots, N$

$$\vec{\Phi}_k \circ f_k^i \to \vec{\xi}_\infty^i \quad \text{uniformly in } C^0\left(\mathbb{S}^2 \setminus \bigcup_{j=1}^{N^i} B_{s_k}(b^{i,j})\right), \tag{6.3}$$

and

$$\int_{\mathbb{S}^2 \setminus \bigcup_{j=1}^{N^i} B_{s_k}(b^{i,j})} 1\, d\mathrm{vol}_{g_{\vec{\Phi}_k \circ f_k^i}} \to \int_{\mathbb{S}^2} 1\, d\mathrm{vol}_{g_{\vec{\xi}^i_\infty}} = \mathrm{Area}(\vec{\xi}^i_\infty(\mathbb{S}^2)). \tag{6.4}$$

Denote

$$S_k^i := \mathbb{S}^2 \setminus \bigcup_{j=1}^{N^i} B_{s_k}(b^{i,j}). \tag{6.5}$$

For any $i' \neq j$ there exists an $i \in \{1, \dots, N^i\}$ such that

$$(f_k^i)^{-1} \circ f_k^{i'}(S_k^{i'}) \subset B_{s_k}(b^{i,j}). \tag{6.6}$$

For each $i \in \{1, \dots, N\}$ and for each $j \in \{1, \dots, N^i\}$ we denote by $J^{i,j}$ the set of indices i' such that (6.6) *holds. We have finally*

$$\lim_{k\to\infty} \mathrm{diam}\left(\vec{\Phi}_k \circ f_k^i \left(B_{s_k}(b^{i,j}) \setminus \bigcup_{i' \in J^{i,j}} \mathrm{Conv}\big((f_k^i)^{-1} \circ f_k^{i'}(S_k^{i'})\big)\right)\right) = 0 \tag{6.7}$$

for all $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, N^i\}$, where $\mathrm{Conv}(X)$ *denotes the convex hull of a set X, and*

$$\lim_{k\to\infty} \mathrm{Area}\left(\vec{\Phi}_k \circ f_k^i \left(B_{s_k}(b^{i,j}) \setminus \bigcup_{i' \in J^{i,j}} \mathrm{Conv}\big((f_k^i)^{-1} \circ f_k^{i'}(S_k^{i'})\big)\right)\right) = 0 \tag{6.8}$$

for all $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, N^i\}$.

Proof. We construct the f_k^i, the $\vec{\xi}_i^\infty$ and the $b^{i,j}$ by induction. We first apply the "Good Gauge Extraction Lemma" 4.1 and we generate a first sequence f_k^1 of elements of $\mathcal{M}^+(\mathbb{S}^2)$ and a first element $\vec{\xi}^1_\infty \in \mathcal{F}_{\mathbb{S}^2}$ as well as a first collection of points $\{b^{1,1}, \dots, b^{1,N^1}\}$ such that

$$\vec{\xi}_k^1 := \vec{\Phi}_k \circ f_k^1 \rightharpoonup \vec{\xi}^1_\infty \quad \text{weakly in } W^{2,2}(\mathbb{S}^2 \setminus \{b^{1,1}, \dots, b^{1,N^1}\}).$$

Since the weak convergence in $W^{2,2}(\mathbb{S}^2 \setminus \{b^{1,1}, \dots, b^{1,N^1}\})$ implies a strong C^0 convergence of $\{\vec{\xi}_k^1\}_{k\in\mathbb{N}}$ on any compact of $S^1_\infty := \mathbb{S}^2 \setminus \{b^{1,1}, \dots, b^{1,N^1}\}$ but also a strong L^2 convergence of the conformal factors $\{e^{\lambda_k^1}\}_{k\in\mathbb{N}}$ satisfying

$$g_{\vec{\xi}_k^1} = e^{2\lambda_k^1}\, g_{\mathbb{S}^2},$$

and moreover since the limit $\vec{\xi}^1_\infty \in \mathcal{F}_{\mathbb{S}^2}$ is Lipschitz all over $\mathbb{S}^2$, one can always find $t_k \to 0$ such that

$$\vec{\xi}_k^1 := \vec{\Phi}_k \circ f_k^1 \to \vec{\xi}^1_\infty \quad \text{uniformly in } C^0\left(\mathbb{S}^2 \setminus \bigcup_{j=1}^{N^1} B_{t_k}(b^{1,j})\right), \tag{6.9}$$

and such that

$$\int_{\mathbb{S}^2\setminus\bigcup_{j=1}^{N^i} B_{t_k}(b^{i,j})} 1\, d\mathrm{vol}_{g_{\vec{\Phi}_k\circ f_k^1}} \to \int_{\mathbb{S}^2} 1\, d\mathrm{vol}_{g_{\vec{\xi}_\infty^1}} = \mathrm{Area}(\vec{\xi}_\infty^1(\mathbb{S}^2)). \quad (6.10)$$

Consider now for any $j \in \{1, \dots, N^1\}$ and for $l \in \mathbb{N}$ the annuli

$$A_k^{j,l} := B_{t_k^{\frac{1}{l+1}}}(b^{1,j}) \setminus B_{t_k^{\frac{1}{l}}}(b^{1,j}).$$

Because of (6.1), one can find $l_k \in \mathbb{N}$ such that

$$t^{\frac{1}{l_k+1}} \to 0,$$

$$\varepsilon_k^1 := t_k^{\frac{1}{l_k}-\frac{1}{l_k+1}} \to 0$$

and

$$\lim_{k\to+\infty} \int_{A^{j,l_k}} \left[1 + |\vec{H}_{\vec{\Phi}_k}|_h^2\right] d\mathrm{vol}_{g_k} = 0 \quad \text{for all } j.$$

We then choose $\delta_k^1 := t^{\frac{1}{l_k^j+1}}$.

In case for all $j = 1, \dots, N^1$ one has

$$\lim_{k\to 0} \mathrm{diam}\Big(\vec{\Phi}_k \circ f_k^1\big(B_{\delta_k^1}(b^{1,j})\big)\Big) = 0, \quad (6.11)$$

then we stop the procedure at this stage. If not, we take each of the $B_{\delta_k^1}(b^{1,j})$ such that (6.11) does not happen and for each of them we apply the "Diameter Tracking Procedure Lemma" 5.1 with $a := b^{1,j}$, $\delta_k := \delta_k^1$ and $\varepsilon_k := \varepsilon_k^1$ and no a_k^i first. We generate then new f_k^i and observe that each such generation costs an amount of energy which is at least

$$\inf_{\vec{\xi}\in\mathcal{F}_{\mathbb{S}^2}} L(\vec{\xi}) = c_0 > 0.$$

Thus the procedure must stop after finitely many iterations and Lemma 6.1 will be proved once (6.8) will be established.

Proof of (6.8). There is a sequence of conformal transformations φ_k in $\mathcal{M}^+(\mathbb{S}^2)$ such that

$$\varphi_k^{-1}\left(B_{s_k}(b^{i,j}) \setminus \bigcup_{i'\in J^{i,j}} \mathrm{Conv}\big((f_k^i)^{-1} \circ f_k^{i'}(S_k^{i'})\big)\right) = \mathbb{S}^2 \setminus \bigcup_{l=1}^{N^{i,j}+1} B_{s_k^l}(a_k^l)$$

and there exists $\varepsilon_k \to 0$ such that $\varepsilon_k^{-1} s_k^l \to 0$ for any l and

$$\lim_{k\to+\infty} \int_{B_{\varepsilon_k^{-1}s_k^l}(a_k^l)\setminus B_{s_k^l}(a_k^l)} \left[1 + |\mathbb{I}_{\vec{\Phi}_k\circ f_k^i\circ\varphi_k}|^2\right] d\mathrm{vol}_{g_{\vec{\Phi}_k\circ f_k^i\circ\varphi_k}} = 0.$$

To each l we apply the "Cutting and Filling Lemma" 5.2 for $a = a_k^l$, $s_k = \varepsilon_k^{-1} s_k^l$ and $t_k = s_k^l$. We obtain a new conformal immersion $\vec{\zeta}_k^i$ such that

$$d_k^i := \operatorname{diam}(\vec{\zeta}_k^i(\mathbb{S}^2)) \to 0, \quad \limsup_{k\to+\infty} G(\vec{\zeta}_k^i) < +\infty \tag{6.12}$$

and such that for all k

$$\vec{\Phi}_k \circ f_k^i \left(B_{s_k}(b^{i,j}) \setminus \bigcup_{i' \in J^{i,j}} \operatorname{Conv}\big((f_k^i)^{-1} \circ f_k^{i'}(S_k^{i'})\big) \right) \subset \vec{\zeta}_k^i(\mathbb{S}^2). \tag{6.13}$$

By viewing M^m as being isometrically embedded in a Euclidean space $\mathbb{R}^n$, (6.12) together with (3.2) imply that there exists $p_k \in \mathbb{R}^n$ such that

$$\limsup_{k\to+\infty} \int_{\mathbb{S}^2} \big[|D\vec{n}_{\vec{I}\circ\vec{\zeta}_k}|^2\big]\, d\mathrm{vol}_{\vec{I}\circ\vec{\zeta}_k} < +\infty \quad \text{and} \quad \vec{I}\circ\vec{\zeta}_k^i(\mathbb{S}^2) \subset B_{d_k}(p_k). \tag{6.14}$$

By using L. Simon's monotonicity formula [24], we deduce that

$$\limsup_{k\to+\infty} d_k^{-2} \operatorname{Area}\big(\vec{\zeta}_k^i(\mathbb{S}^2)\big) = \limsup_{k\to+\infty} d_k^{-2} \operatorname{Area}\big(\vec{\zeta}_k^i(\mathbb{S}^2) \cap B_{d_k}(p_k)\big) < +\infty. \tag{6.15}$$

Thus we infer that

$$\begin{aligned} &\operatorname{Area}\left(\vec{\Phi}_k \circ f_k^i \left(B_{s_k}(b^{i,j}) \setminus \bigcup_{i' \in J^{i,j}} \operatorname{Conv}\big((f_k^i)^{-1} \circ f_k^{i'}(S_k^{i'})\big) \right)\right) \\ &\quad \le \operatorname{Area}\big(\vec{\zeta}_k^i(\mathbb{S}^2)\big) = O(d_k^2) \to 0. \end{aligned} \tag{6.16}$$

This implies (6.8) and Lemma 6.1 is finally proved. □

Proof of Theorem 1.5. We construct Ψ_k step by step in the following way.

We consider in $\mathbb{S}^2$ N^1 disjoint fixed balls $(B^{1,j})_{j=1,\dots,N^1}$ and we consider a sequence of diffeomorphisms Λ_k^1,

$$\Lambda_k^1 : \mathbb{S}^2 \setminus \bigcup_{j=1}^{N^1} B_{s_k}(b^{1,j}) \to \mathbb{S}^2 \setminus \bigcup_{j=1}^{N^1} B^{1,j},$$

such that

$$\Lambda_k^1(\partial B_{s_k}(b^{1,j})) = \partial B^{1,j}$$

and

$$\limsup_{k\to+\infty} \|\nabla(\Lambda_k^1)^{-1}\|_{L^\infty(\mathbb{S}^2 \setminus \bigcup_{j=1}^{N^1} B^{1,j})} < +\infty, \tag{6.17}$$

and such that $\Xi_k^1 := (\Lambda_k^1)^{-1}$ converges weakly in $(W^{1,\infty})^*$ to a limiting diffeomorphism

$$\Xi_\infty^1 : \mathbb{S}^2 \setminus \bigcup_{j=1}^{N^1} \overline{B^{1,j}} \to \mathbb{S}^2 \setminus \{b^{1,1}, \dots, b^{1,N^1}\}.$$

We adopt the same notation as in the proof of Lemma 6.1: for any $k \in \mathbb{N}$ and $i = 1, \dots, N$

$$S_k^i := \mathbb{S}^2 \setminus \bigcup_{j=1}^{N^i} B_{s_k}(b^{i,j}),$$

and $J^{i,j}$ is the following set of indices:

$$J^{i,j} := \left\{ i' : (f_k^i)^{-1} \circ f_k^{i'}(S_k^{i'}) \subset B_{s_k}(b^{i,j}) \right\};$$

we also let $\hat{J}^{i,j}$ be the following subset of $J^{i,j}$

$$\hat{J}^{i,j} := \{ i' \in J^{i,j} : \nexists i'' \in J^{i,j} \text{ such that } \\ (f_k^i)^{-1} \circ f_k^{i'}(S_k^{i'}) \subset \mathrm{Conv}\big((f_k^i)^{-1} \circ f_k^{i''}(S_k^{i''})\big)\big\},$$

where $\mathrm{Conv}(X)$ is the convex hull of X in $\mathbb{S}^2$. We denote by $N^{i,j}$ the cardinality of $\hat{J}^{i,j}$. Recall that, due to (6.7), one has for any $i \in \{1, \dots, N\}$ and any $j \in \{1, \dots, N^{i,j}\}$

$$\lim_{k \to 0} \mathrm{diam}\left(\vec{\Phi}_k \circ f_k^i \left(B_{s_k}(b^{i,j}) \setminus \bigcup_{i' \in \hat{J}^{i,j}} \mathrm{Conv}\big((f_k^i)^{-1} \circ f_k^{i'}(S_k^{i'})\big) \right) \right) = 0. \quad (6.18)$$

For any $j \in \{1, \dots, N^1\}$, inside $B^{1,j}$, we fix $N^{1,j}$ disjoint balls independent of k. We denote each of this ball by B^i for each $i \in \hat{J}^{i,j}$. We can always reorder the indexation in such a way that

$$(f_k^i)^{-1} \circ f_k^1\left(S_k^1\right) \subset B_{s_k}(b^{i,N_i}).$$

In each of the B^i we fix again $N^i - 1$ disjoint balls $(B^{i,j})_{j=1,\dots,N^i-1}$ each being independent of k. For each of these $i \in \hat{J}^{1,j}$ we pick a sequence of diffeomorphisms Λ_k^i,

$$\Lambda_k^i : \mathbb{S}^2 \setminus \bigcup_{j=1}^{N^i} B_{s_k}(b^{i,j}) \to B^i \setminus \bigcup_{j=1}^{N^i-1} B^{i,j},$$

such that

$$\Lambda_k^i(\partial B_{s_k}(b^{i,j})) = \partial B^{i,j} \quad \text{for all } j = 1, \dots, N^i - 1,$$

$$\Lambda_k^i(\partial B_{s_k}(b^{i,N^i})) = \partial B^i,$$

and

$$\limsup_{k\to+\infty} \|\nabla(\Lambda_k^i)^{-1}\|_{L^\infty(B^i\setminus\bigcup_{j=1}^{N^i-1} B^{i,j})} < +\infty, \tag{6.19}$$

and such that $\Xi_k^i := (\Lambda_k^i)^{-1}$ converges weakly in $(W^{1,\infty})^*$ to a limiting diffeomorphism

$$\Xi_\infty^i : \mathbb{S}^2 \setminus \bigcup_{j=1}^{N^i} \overline{B^{i,j}} \to \mathbb{S}^2 \setminus \{b^{i,1}, \dots, b^{i,N^i}\}.$$

We iterate this procedure in a straightforward way until having exhausted all the indices $i \in \{1, \dots, N\}$.

We now fix the sequence of diffeomorphism Ψ_k in the following way:

(i) $\Psi_k := f_k^1 \circ \Xi_k^1$ on $\mathbb{S}^2 \setminus \bigcup_{j=1}^{N^1} B^{1,j}$,

(ii) $\Psi_k := f_k^i \circ \Xi_k^i$ on $B^i \setminus \bigcup_{j=1}^{N^i-1} B^{i,j}$,

(iii) Ψ_k is chosen arbitrarily among the diffeomorphisms from

$$B^{i,j} \setminus \bigcup_{i'\in\hat{J}^{i,j}} B^{i'}$$

into

$$f_k^i\big(B_{s_k}(b^{i,j})\big) \setminus \bigcup_{i'\in\hat{J}^{i,j}} f_k^{i'}\big(B_{s_k}(b^{i',N^{i'}})\big)$$

such that $\Psi_k := f_k^i \circ \Xi_k^i$ on $\partial B^{i,j}$ and $\Psi_k := f_k^{i'} \circ \Xi_k^{i'}$ on $\partial B^{i'}$.

Because of (6.18), we have that, modulo extraction of a subsequence, for any $i > 1$ and $j = 1, \dots, N^i - 1$, there exists a point $z^{i,j} \in M^m$ such that

$$\vec{\Phi}_k \circ \Psi_k \to p^{i,j} \quad \text{uniformly in } C^0\Bigg(B^{i,j} \setminus \bigcup_{i'\in\hat{J}^{i,j}} B^{i'}\Bigg). \tag{6.20}$$

Observe that

$$\Xi_k^1 \rightharpoonup \Xi_\infty^1 \quad \text{weakly in } (W^{1,\infty})^*\Bigg(\mathbb{S}^2 \setminus \bigcup_{j=1}^{N^1} B^{1,j}\Bigg),$$

and for $i \geq 2$

$$\Xi_k^i \rightharpoonup \Xi_\infty^i \quad \text{weakly in } W^{1,\infty}\Bigg(B^i \setminus \bigcup_{j=1}^{N^i-1} B^{i,j}\Bigg).$$

Define now $\vec{f}_\infty$ from $\mathbb{S}^2$ into M^m by

$$\vec{f}_\infty := \vec{\xi}^{\,1}_\infty \circ \Xi^1_\infty \quad \text{on } \mathbb{S}^2 \setminus \bigcup_{j=1}^{N^1} B^{1,j},$$

$$\vec{f}_\infty := \vec{\xi}^{\,i}_\infty \circ \Xi^i_\infty \quad \text{on } B^i \setminus \bigcup_{j=1}^{N^i-1} B^{i,j},$$

$$\vec{f}_\infty :\equiv p^{i,j} \qquad \text{on } B^{i,j} \setminus \bigcup_{i' \in \hat{J}^{i,j}} B^{i'}.$$

It is now straightforward to check that the $N+1$-uplet $(\vec{f}_\infty, \vec{\xi}^{\,1}_\infty, \ldots, \vec{\xi}^{\,N}_\infty)$ satisfy the conclusions of Theorem 1.5. This concludes the proof. □

7 Weak closure of bubble trees of weak immersions

As it is, Theorem 1.5 is a weak-semi-closure result and not a weak-closure result *per se* in the sense that the weak limit does not anymore belong to the same class of *weak immersions*, $\mathcal{F}_{\mathbb{S}^2}$, but is made of a finite family of elements of $\mathcal{F}_{\mathbb{S}^2}$ that we will call a *bubble tree of weak immersions*. In order to remedy to this difficulty we are going to define rigorously the class of *bubble tree of weak immersions* and prove afterwards a weak closure result in this class (see Theorem 7.2).

Definition 7.1 (Bubble trees of weak immersions). We call a *bubble tree of weak immersions* an $N+1$-tuple $\vec{T} := (\vec{f}, \vec{\Phi}^1, \ldots, \vec{\Phi}^N)$, where N is an arbitrary integer, $\vec{f} \in W^{1,\infty}(\mathbb{S}^2, M^m)$ and $\vec{\Phi}^i \in \mathcal{F}_{\mathbb{S}^2}$ for $i = 1, \ldots, N$ satisfy the following conditions. There exists a family of N geodesic balls $B^i \subset \mathbb{S}^2$ such that

- for all $i \neq i'$ either $\overline{B^i} \subset B^{i'}$ or $\overline{B^{i'}} \subset B^i$ and $B^1 = \mathbb{S}^2$.

For any $i \in \{1, \ldots, N\}$ there exist a natural integer N^i and N^i disjoint open geodesic balls $B^{i,j}$ strictly included in B^i such that

- for all $i' \neq i$ either $\overline{B^i} \subset B^{i'}$ or there exists an $j \in \{1, \ldots, N^i\}$ such that $\overline{B^{i'}} \subset B^{i,j}$.

For any $i \in \{1, \ldots, N\}$ there exist N^i distinct points $b^{i,1}, \ldots, b^{i,N^i}$ of $\mathbb{S}^2$ and a Lipschitz diffeomorphism

$$\Xi^i : B^i \setminus \bigcup_{j=1}^{N^i-1} \overline{B^{i,j}} \to \mathbb{S}^2 \setminus \{b^{i,1}, \ldots, b^{i,N^i}\}$$

such that Ξ_i extends to a Lipschitz map

$$\overline{\Xi}_i : \overline{B^i} \setminus \bigcup_{j=1}^{N^i-1} B^{i,j} \to \mathbb{S}^2$$

such that

$$\overline{\Xi}_i(\partial B^{i,j}) = b^{i,j} \quad \text{and} \quad \overline{\Xi}_i(\partial B^i) = b^{i,N^i}.$$

Moreover for all $i = 1, \ldots, N$ one has

$$\vec{f} = \vec{\Phi}^i \circ \Xi^i \quad \text{on } B^i \setminus \bigcup_{j=1}^{N^i-1} B^{i,j}$$

and for any $i \in \{1, \ldots, N\}$ and any $j \in \{1, \ldots, N^i\}$ there exists a point $p^{i,j} \in M^m$ such that

$$\vec{f} :\equiv p^{i,j} \quad \text{on } B^{i,j} \setminus \bigcup_{i' \in \hat{J}^{i,j}} B^{i'},$$

where

$$J^{i,j} := \{i' : \overline{B^{i'}} \subset B^{i,j}\}.$$

We denote by $\mathcal{T}$ the space of bubble trees of weak immersions, and for any $\vec{T} \in \mathcal{T}$ we denote

$$G(\vec{T}) := \sum_{i=1}^{N} G(\vec{\Phi}^i) := \sum_{i=1}^{N} A(\vec{\Phi}^i) + F(\vec{\Phi}^i) = \sum_{i=1}^{N} \int_{\mathbb{S}^2} \left[1 + \frac{|D\vec{n}_{\vec{\Phi}^i}|^2}{2}\right] d\,\mathrm{vol}_{g_{\vec{\Phi}^i}}.$$

Assuming $x_0 \in \vec{f}(\mathbb{S}^2)$, the homotopy class of $\vec{T}$ in $\pi_2(M^m, x_0)$ is the class of $\vec{f}$ and is denoted by $[\vec{T}]$.

The subspace of elements $\vec{T}$ in $\mathcal{T}$ such that $x_0 \in \vec{f}(\mathbb{S}^2)$ is denoted by $\mathcal{T}_{x_0}$.

Theorem 7.2 (Weak closure of bubble trees of weak immersions). *Let*

$$T_k = (\vec{f}_k, \vec{\Phi}_k^1, \ldots, \vec{\Phi}_k^{N_k})$$

be a sequence of elements in $\mathcal{T}$ such that

$$\limsup_{k\to+\infty} G(\vec{T}_k) = \limsup_{k\to+\infty} \sum_{i=1}^{N_k} \int_{\mathbb{S}^2} \left[1 + \frac{|D\vec{n}_{\vec{\Phi}_k^i}|^2}{2}\right] d\,\mathrm{vol}_{g_{\vec{\Phi}_k^i}} < +\infty.$$

Assume each $\vec{\Phi}_k^i$ is weakly conformal and that

$$\liminf_{k\to+\infty} \sum_{i=1}^{N_k} \mathrm{diam}(\vec{\Phi}_k^i(\mathbb{S}^2)) > 0.$$

Then there exists a subsequence that we keep denoting $\vec{T}_k$ such that $N_k = N$ is constant and there exists a sequence of Lipschitz diffeomorphisms Ψ_k of $\mathbb{S}^2$ such that

$$\vec{f}_k \circ \Psi_k \to \vec{u}_\infty \quad \textit{uniformly in } C^0(\mathbb{S}^2, M^m), \tag{7.1}$$

where $\vec{u}_\infty \in W^{1,\infty}(\mathbb{S}^2, M^m)$, such that

$$\mathrm{Area}(\vec{f}_k(\mathbb{S}^2)) \to \mathrm{Area}(\vec{u}_\infty(\mathbb{S}^2)); \tag{7.2}$$

moreover for any $i = 1, \ldots, N$ there exist $Q^i \in \mathbb{N}$ and Q^i sequences of elements

$$f_k^{i,j} \in \mathcal{M}^+(\mathbb{S}^2)$$

and for each (i, j) there exist finitely many points $b^{i,j,1}, \ldots, b^{i,j,Q^{i,j}}$ such that

$$\vec{\Phi}_k^i \circ f_k^{i,j} \rightharpoonup \vec{\xi}_\infty^{i,j} \quad \textit{weakly in } W^{2,2}_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{b^{i,j,1}, \ldots, b^{i,j,Q^{i,j}}\}). \tag{7.3}$$

The maps $\vec{\xi}_\infty^{i,j}$ are conformal weak immersions of $\mathcal{F}_{\mathbb{S}^2}$ and

$$\left(\vec{u}_\infty, (\vec{\xi}_\infty^{1,j})_{j=1,\ldots,Q^1}, \ldots, (\vec{\xi}_\infty^{N,j})_{j=1,\ldots,Q^N}\right) \in \mathcal{T}. \tag{7.4}$$

Proof. First of all the uniform bound on $G(\vec{T}_k)$ implies that N_k is uniformly bounded as well as all the numbers N_k^i associated to the underlying tree. We can then extract a subsequence such that these numbers are uniformly bounded and such that the associated tree $(i, j) \to J^{i,j}$ is fixed. We can moreover find a sequence of diffeomorphisms Ψ_k such that, replacing $\vec{f}_k$ by $\vec{f}_k \circ \Psi_k$, the B^i, the $B^{i,j}$, the Ξ^i and the $b^{i,j}$ are fixed independently of k and satisfy for all $i = 1, \ldots, N$

$$\vec{f}_k \circ \Psi_k = \vec{\Phi}_k^i \circ \Xi^i \quad \text{on } B^i \setminus \bigcup_{j=1}^{N^i-1} B^{i,j}.$$

Finally, since M^m is compact, we can also choose the subsequence such that

$$\vec{f}_k \circ \Psi_k = p_k^{i,j} \to p_\infty^{i,j} \in M^m \quad \text{on } B^{i,j} \setminus \bigcup_{i' \in \hat{J}^{i,j}} B^{i'}.$$

Now we can apply Theorem 1.5 to each of the sequences $\vec{\Phi}_k^i$. Modifying accordingly the diffeomorphism Ψ_k in each of the $B^{i,j} \setminus \bigcup_{i' \in \hat{J}^{i,j}} B^{i'}$ and collecting all information together provides (7.1), (7.3) and (7.4); Theorem 7.2 is hence proved. $\square$

The weak closure of bubble trees of weak immersions, namely Theorem 7.2, implies in a straightforward way the following corollary once it is known that any homotopy class $\pi_2(M, x_0)$ can be realized by an element in $\mathcal{F}_{\mathbb{S}^2}$.

Corollary 7.3. *Let $x_0 \in M^m$ and let $\gamma \in \pi_2(M^m, x_0)$. Assume $\gamma \neq 0$. Then*

$$\inf_{\substack{\vec{T} \in \mathcal{T}_{x_0} \\ [T]=\gamma}} G(\vec{T})$$

is achieved by an element $\vec{T}_\gamma := (\vec{f}_\gamma, \vec{\Phi}^1_\gamma, \dots, \vec{\Phi}^N_\gamma)$. Moreover if we do not fix the base point and consider now $\gamma \in \pi_2(M^m)$ and $\gamma \neq 0$, we have that

$$\inf_{\substack{\vec{T} \in \mathcal{T} \\ [T]=\gamma}} G(\vec{T})$$

is achieved by an element $\vec{G}_\gamma := (\vec{f}_\gamma, \vec{\Phi}^1_\gamma, \dots, \vec{\Phi}^N_\gamma)$.

We will show in [16] that, for any $\gamma \in \pi_2(M^m)$, the $\vec{\Phi}^i_\gamma$ realize *area constrained smooth, possibly branched, Willmore immersions* of $\mathbb{S}^2$.

Proof. We just have to prove that for any $\gamma \in \pi_2(M^m, x_0)$ there exists an element of $\mathcal{F}_{\mathbb{S}^2}$ realizing γ. From Theorem 1.1 we have a family of possibly branched conformal smooth immersions $(\vec{\Phi}^i)_{i=1,\dots,N}$ which generates $\pi_2(M^m)$ modulo the action of $\pi_1(M^m)$. In other words we can connect these immersions by tubular neighbourhoods of C^1 paths going either from a given base point to these branched immersed spheres or from one of these spheres to another one in order to have a generating family of $\pi_2(M^m)$. There is no difficulty to "connect" these spheres by these tubes in order to realize a branched immersion of $\mathcal{F}_{\mathbb{S}^2}$. We have then proved that any $\gamma \in \pi_2(M^m, x_0)$ can be realized by an element of $\mathcal{F}_{\mathbb{S}^2}$. This fact combined with Theorem 7.2 implies Corollary 7.3. □

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